Finiteness and existence of attractors and repellers on sectional hyperbolic sets

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Preliminary definitions

Consider a compact Riemannian manifold M of dimension $n \geq 3$ (a *compact n-manifold* for short). We denote by ∂M the boundary of M. Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denotes by X_t the flow of X, $t \in \mathbb{R}$.
Given the Riemannian metric $\langle *, * \rangle$ on M, one has a $2-$Norm $\|*, *\|$ such that for $x \in M$:

$$\|*, *\|_x : T_xM \times T_xM \to \mathbb{R}^+ \cup \{0\}$$

$$(u, v) \mapsto \|u, v\|_x = \sqrt{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}$$

which measure the area of the parallelogram spanned by the vectors u and v on T_xM.

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- *Attracting set* if $\Lambda = \cap_{t>0} X_t(U)$ for some compact neighborhood U of Λ. This neighborhood is called *isolating block* of Λ.

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- An *Attractor* is a transitive attracting set. An attractor is *nontrivial* if it is not a closed orbit.
- A *repelling* is an attracting for the time reversed vector field $-X$ and a *repeller* is a transitive repelling set.

General definitions

The *maximal invariant* set of X is defined by $M(X) = \bigcap_{t \geq 0} X_t(M)$.

Hyperbolic set

Definition

A compact invariant set Λ of X is hyperbolic *if there are a continuous invariant splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that*

- *E^X is the subespace generated by $X(x)$ in $T_x M$, for every $x \in \Lambda$.*

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- E^X *is the subespace generated by* $X(x)$ *in* $T_x M$*, for every* $x \in \Lambda$*.*
- E^s *is* contracting, *i.e.,* $\| DX_t(x) v_x^s \| \leq C e^{-\lambda t} \| v_x^s \|$*, for all* $x \in \Lambda$*,* $v_x^s \in E_x^s$ *and* $t \geq 0$*.*

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- E^u *is* expanding *, i.e.,* $\| DX_t(x) v_x^u \| \leq C^{-1} e^{\lambda t} \| v_x^u \|$*, for all* $x \in \Lambda$*,* $v_x^u \in E_x^u$ *and* $t \geq 0$*.*

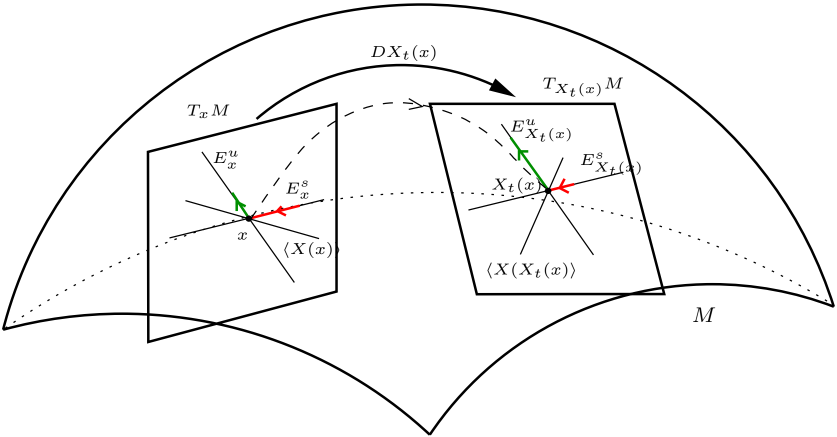

Figure : Hyperbolicity property.

Dominated splitting

Definition

A compact invariant set Λ *of* X *has a* dominated splitting with respect to the tangent flow *if there is an invariant splitting* $T_\Lambda M = E \oplus F$ *such that the following property holds for some positive constants* C, λ*:*

$$|| \, DX_t(x)e_x \, || \cdot || \, f_x \, || \leq Ce^{-\lambda t} \, || \, DX_t(x)f_x \, || \cdot || \, e_x \, ||,$$

for all $x \in \Lambda$*,* $(e_x, f_x) \in E_x \times F_x$ *and* $t \geq 0$*.*

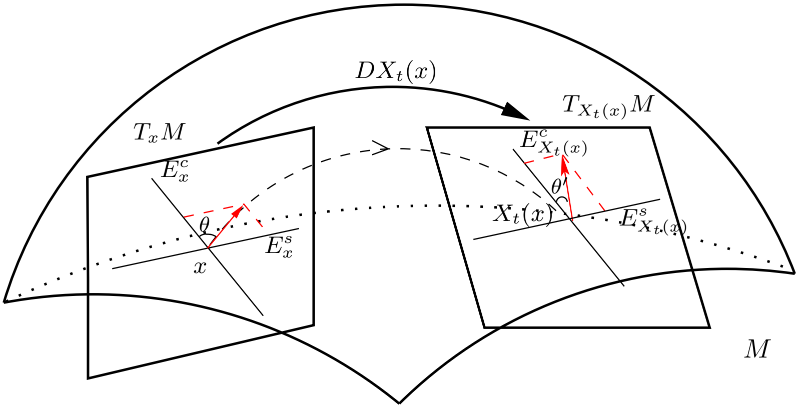

Figure : Domination property.

Partially hyperbolic set

Definition

A compact invariant set Λ of X is partially hyperbolic *if there is a continuous invariant splitting $T_\Lambda M = E^s \oplus E^c$ such that the following properties hold for some positive constants C, λ:*

- E^s *is* contracting, *i.e.,*

$$|| DX_t(x)v_x^s || \leq Ce^{-\lambda t} || v_x^s ||,$$

for all $x \in \Lambda$, $v_x^s \in E_x^s$ and $t \geq 0$.

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$$\| DX_t(x)v_x^s \| \leq Ce^{-\lambda t} \| v_x^s \|,$$

  for all $x \in \Lambda$, $v_x^s \in E_x^s$ *and* $t \geq 0$.
- E^s dominates E^c.

Sectional hyperbolic set

Definition

A compact invariant set Λ *of* X *is a* sectional hyperbolic set *whose singularities (if any) are hyperbolic if there are a continuous tangent bundle invariant decomposition* $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$, *positive constants* K, λ *and Riemannian metric* $\langle *, * \rangle$ *such that for each* x *in* Λ *and every* $t \geq 0$:

- $\left\| DX_t(x)|_{E^s_x} \right\| \leq Ke^{-\lambda t}$;

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- $\left\| DX_t(x)|_{E^s_x} \right\| \leq Ke^{-\lambda t}$;
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- $\left\| DX_t(x)|_{E^s_x} \right\| \cdot \left\| DX_{-t}(X_t(x))|_{E^c_{X_t(x)}} \right\| \leq Ke^{-\lambda t}$;
- $\left\| DX_t(x)u, DX_t(x)v \right\|_{X_t(x)} \geq K^{-1} e^{\lambda t} \left\| u, v \right\|_x, \ \forall u, v \in E^c_x$.

Sectional Anosov flow

Definition

We say that X is a sectional Anosov flow *if $M(X)$ is a sectional hyperbolic set.*

Stable and instable manifold

$$
\begin{aligned}
W^{ss}_X(p) &= \{x : d(X_t(x), X_t(p)) \to 0, t \to \infty\}, \\
W^{uu}_X(p) &= \{x : d(X_t(x), X_t(p)) \to 0, t \to -\infty\},
\end{aligned}
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$$\begin{aligned} W^{s}_X(p) &= \textstyle\bigcup_{t\in\mathbb{R}} W^{ss}_X(X_t(p)), \\ W^{u}_X(p) &= \textstyle\bigcup_{t\in\mathbb{R}} W^{uu}_X(X_t(p)). \end{aligned}$$

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$$\begin{aligned} W^{ss}_X(p,\epsilon) &= \{x : d(X_t(x), X_t(p)) \le \epsilon, \forall t \ge 0\}, \\ W^{uu}_X(p,\epsilon) &= \{x : d(X_t(x), X_t(p)) \le \epsilon, \forall t \le 0\} \end{aligned}$$

Lorenz-like singularity

Definition

We say that a singularity σ of a sectional-Anosov flow X is Lorenz-like *if* $dim(W^s(\sigma)) = dim(W^{ss}(\sigma)) + 1$.

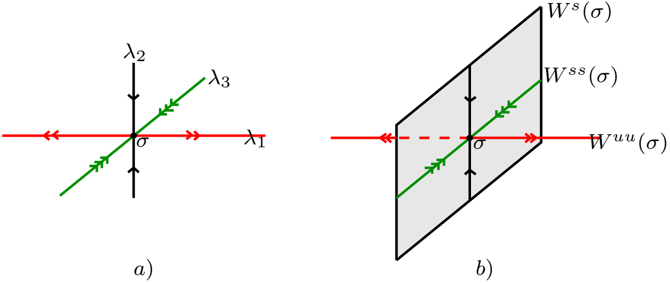
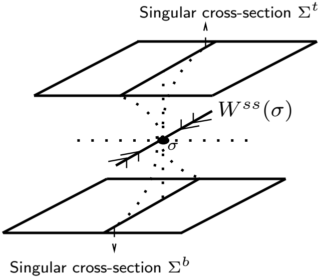


Figure : Three dimensional Lorenz-Like singularity.

Singular-cross section



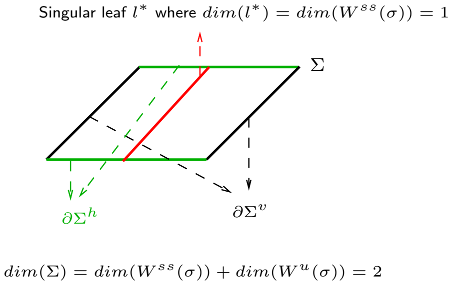


Figure : Three dimensional case. Cross-section.

Singular-cross section

Definition

A singular-cross section *of a Lorenz-like singularity σ will be a pair of submanifolds Σ^t, Σ^b, where Σ^t, Σ^b are cross sections and;*
Σ^t is transversal to $W^{s,t}_{loc}(\sigma)$.
Σ^b is transversal to $W^{s,b}_{loc}(\sigma)$.

Note that every singular-cross section contains a pair singular submanifolds l^t, l^b defined as the intersection of the local stable manifold of σ with Σ^t, Σ^b respectively and $dim(l^*) = dim(W^{ss}(\sigma))$, with (*=t,b).

Singular-cross section

Σ^* is a *hipercube of dimension* $(n-1)$, diffeomorphic to $B^u[0,1] \times B^{ss}[0,1]$, with $B^u[0,1] \approx I^u$, $B^{ss}[0,1] \approx I^s$, $I^k = [-1,1]^k$, $k \in \mathbb{Z}$ and where:

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$B^{ss}[0,1]$ is a ball centered at zero and radius 1 contained in $\mathbb{R}^s$
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Let $f : B^u[0,1] \times B^{ss}[0,1] \longrightarrow \Sigma^*$ be the diffeomorphism, such that $f(\{0\} \times B^{ss}[0,1]) = l^*$ and $\{0\} = 0 \in \mathbb{R}^u$.

Singular-cross section

Let $\partial\Sigma^*$ be the boundary of Σ^*, and $\partial\Sigma^* = \partial^h\Sigma^* \cup \partial^v\Sigma^*$ where
$\partial^h\Sigma^* = \{$ Union of the boundary submanifolds transverse to l^* $\}$
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Moreover,

$$\partial^h\Sigma^* = (I^u \times [\cup_{j=0}^{s-1}(I^j \times \{-1\} \times I^{s-j-1})]) \bigcup (I^u \times [\cup_{j=0}^{s-1}(I^j \times \{1\} \times I^{s-j-1})$$

$$\partial^v\Sigma^* = ([\cup_{j=0}^{u-1}(I^j \times \{-1\} \times I^{u-j-1})] \times I^s) \bigcup ([\cup_{j=0}^{u-1}(I^j \times \{1\} \times I^{u-j-1})] \times I$$

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Hereafter we denote $\Sigma^* = B^u[0,1] \times B^{ss}[0,1]$.

Singular-cross section

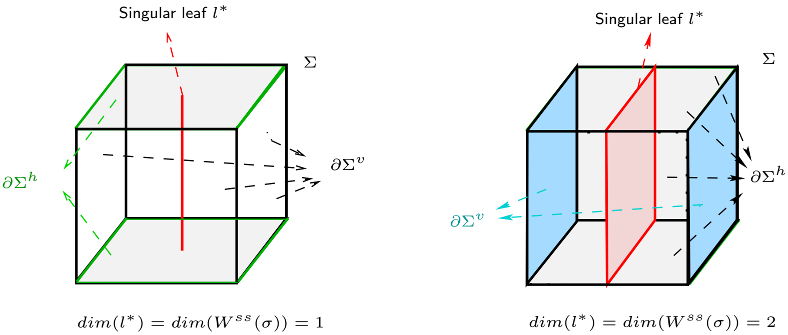

Figure : Four dimensional case. Cross-sections.

MOTIVATION

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- *Brandao, P., Palis, J., Pinheiro, V.*, On the finiteness of attractors for one-dimensional maps with discontinuities (2014) [1]

Main theorem

Theorem (A)

(-) For every sectional hyperbolic set Λ of a vector field X on a compact manifold there are neighborhoods $\mathcal{U}$ of X, U of Λ and $n_0 \in \mathbb{N}$ such that

$$\#\{L \subset U : L \text{ is an attractor or repeller of } Y \in \mathcal{U}\} \leq n_0.$$

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- *BONATTI'S CONJECTURE FOR THREE DIMENSIONAL FLOWS*

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Main theorem

Theorem (B)

(-) Let X be a C^1 vector field with singularities of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a connected sectional hyperbolic set of X. If $\Lambda \subset \Omega(X)$, then there are neighborhoods $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and $U \subset M$ of Λ such that if $Y \in \mathcal{U}$, Y has no repeller in U.

Lemma (Hyperbolic lemma-Morales, C.A. , Pacífico, M.J., Pujals, E.)

Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. Then, there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and a neighborhood $U \subset M$ of Λ such that if $Y \in \mathcal{U}$, every nonempty, compact, non singular, invariant set H of Y in U is hyperbolic saddle-type *(i.e. $E^s \neq 0$ and $E^u \neq 0$).*

Definition

Let A and B be compact sets of M, and let $d(\cdot,\cdot)$ be a metric on M. Define the Hausdorff distance *between A and B by*

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}.$$

We define $K(M) = \{A \subset M \mid A \text{ is compact } \}$.

Remark

We have that d_H is a metric on $K(M)$ and the metric space $(K(M), d_H)$ is compact by Blaschke's selection theorem*.*

Theorem (Lectures of sectional Anosov flows-Bautista, S., Morales, C.A.)

Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. If $\sigma \in Sing(X) \cap \Lambda$, then $\Lambda \cap W_X^{ss}(\sigma) = \{\sigma\}$.

Proposition (-)

Let Λ be a sectional hyperbolic set of a C^1 vector field X of M. Let σ be a Lorenz-like singularity of X in Λ. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Then, there are a neighborhood $U \subset M$ of Λ, a singular-cross section Σ^t, Σ^b of σ in M and $N \in \mathbb{N}$ enough large such that for every $n \geq N$ one has

$$(\Lambda_{Y^n}) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

Corollary

Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of X. Let σ be a singularity of X in Λ. Then, there are $\epsilon > 0$ and a neighborhood V of $W^{ss}(\sigma, \epsilon) \setminus \{\sigma\}$ such that

$$\Lambda \cap V = \emptyset.$$

We define the stable and unstable manifolds of O by

$$W^s(O) = \cup_{x \in O} W^{ss}(x), \text{ and } W^u(O) = \cup_{x \in O} W^{uu}(x)$$

A *homoclinic orbit* of a hyperbolic periodic orbit O is an orbit in $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_qM = T_qW^s(O) + T_qW^u(O)$ for some point $q \in \gamma$, then we say that γ is a *transverse homoclinic orbit* of O.

Definition

The homoclinic class $H(O)$ *of a hyperbolic periodic orbit* O *is the closure of the union of the transverse homoclinic orbits of* O. *We say that an invariant set* L *is a* homoclinic class *if* $L = H(O)$ *for some hyperbolic periodic orbit* O.

Lemma

Let X be a C^1 vector field of a compact n-manifold M, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a hyperbolic set of X. Then, there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X, a neighborhood $U \subset M$ of Λ and $n_0 \in \mathbb{N}$ such that

$$\#\{L \subset U : L \text{ is homoclinic class of } Y \in \mathcal{U}\} \leq n_0$$

for every vector field $Y \in \mathcal{U}$.

Lemma

Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a sectional hyperbolic set of X. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Then, there is a neighborhood $U \subset M$ of Λ, such that if R^n is a repeller of Y^n, $R^n \subset \Lambda_{Y^n}$ for each $n \in \mathbb{N}$, then the sequence $(R^n)_{n \in \mathbb{N}}$ of repellers do not accumulate on the singularities of X, i.e.,

$$Sing(X) \bigcap Cl(\cup_{n \in \mathbb{N}} R^n) = \emptyset$$

Proposition

Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \subset M$ be a sectional hyperbolic set of X. Then, there are neighborhoods $\mathcal{U}$ of X, U of Λ and $n_0 \in \mathbb{N}$ such that

$$\#\{A \subset U : A \text{ is an attractor of } Y \in \mathcal{U}\} \leq n_0.$$

Corollary

For every sectional-Anosov flow of a compact manifold there are a neighborhood $\mathcal{U}$ and $n_0 \in \mathbb{N}$ such that

$$\#\{\ L \text{ is an attractor or repeller of } Y \in \mathcal{U}\} \leq n_0.$$

Lemma

Let X be a C^1 vector field of a compact n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1(M)$. Let $\Lambda \in M$ be a connected sectional hyperbolic set of X with singularities. If $\Lambda \subset \Omega(X)$, then Λ has no repellers.

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