Diffusion through non-transverse heteroclinic chains: A long-time instability for the NLS

Dynamics, Bifurcations, and Strange Attractors

dedicated to 80th birthday of Leonid Pavlovich Shilnikov

Nizhny Novgord, July 20–24, 2015

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Motivation and goals

In [Colliander et al. 10] the authors prove a global instability in the periodic cubic defocusing nonlinear Schrödinger (NLS) equation.

Main Goal: To understand the mechanism that gives rise to this instability and try to generalize to other cases giving a scheme of detection.

J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math. **181** (2010), no.1, 39-113. MR 2651381.

Instability in the periodic cubic defocusing NLS found in [Colliander et al. 10]

$$\begin{cases} -i\partial_t u + \Delta u &= |u|^2 u, \\ u(0,x) &= u_0(x), \quad u(t,x) \in \mathbb{C}, x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2) \end{cases} \tag{1}$$

The main result of this paper is the construction of solutions with arbitrarily large growth in time of the s-Sobolev norm for $s > 1$.

$$\|u(T)\|_{H^s} = \left(\sum_{n\in\mathbb{Z}^2} |n|^{2s} |\hat{u}(t,n)|^2 \right)^{1/2s}$$

Theorem 1 (Colliander et al. 10)

Let $s > 1$, $K \gg 1$ and $0 < \delta \ll 1$ be given parameters. Then there exists a global smooth solution $u(t,x)$ to (1) and a time $T > 0$ with

$$\|u(0)\|_{H^s} \leq \delta \qquad \textit{and} \qquad \|u(T)\|_{H^s} \geq K.$$

Previous works

[Bourgain 00]: Conjectured the existence of solutions of unbounded growth of the Sobolev norm.

[Colliander et al. 10]

[Guardia-Kaloshin 15]: Estimate the diffusing time (introducing typical techniques from the Dynamical Systems).

[Guardia 15]: Estimates the diffusing time when one considers the cubic NLS with a convolution potential.

[Haus-Procesi 14, Guardia-Haus-Procesi 15]: Following [Colliander et al. 10], prove the same result for the quintic NLS and for any odd power in the nonlinearity, respectively.

[Hani 14]: Proves the unboundedness of Sobolev norms in solutions of the resonant truncation of the NLS.

[Hani-Pausader-Tzvetkov-Visciglia 14]: Prove that there exists solutions of the NLS in $\mathbb{R} \times \mathbb{T}^2$ that asymptotically approach solutions of the resonant truncation of the NLS.

Reduction of the problem

PDE (NLS)
$$-i\partial_t u + \Delta u = |u|^2 u$$

Infinite System of ODE's
($\mathcal{F}NLS$)
$$-\mathrm{i}\frac{\mathrm{d}a_n}{\mathrm{d}t} = |n|^2 a_n + \sum_{\Gamma(n)} a_{n_1}\overline{a_{n_2}}a_{n_3}$$
$$\Gamma(n) = \{n_1 - n_2 + n_3 = n\}$$

Finite System
of ODE's
(Toy Model System)

Infinite System of ODE's
($R\mathcal{F}NLS$)
$$-\mathrm{i}\frac{\mathrm{d}r_n}{\mathrm{d}t} = |n|^2 r_n + \sum_{\Gamma_{\mathrm{res}}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3}$$
$$\Gamma_{res}(n) = \Gamma(n) \cap \{|n_1|^2 - |n_2|^2 + |n_3|^2 = |n|^2\}$$

Reduction of the problem

PDE (NLS)
$-i\partial_t u + \Delta u = |u|^2 u$

Fourier
$u(t,x) = \sum a_n(t) e^{in\cdot x}$

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($\mathcal{F}NLS$)

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Reduction of the problem

PDE (NLS)
$$-i\partial_t u + \Delta u = |u|^2 u$$

Fourier
$$u(t,x) = \sum a_n(t) \mathrm{e}^{\mathrm{i} n\cdot x}$$

Infinite System of ODE's
$(\mathcal{F}NLS)$
$$-\mathrm{i}\frac{\mathrm{d}a_n}{\mathrm{d}t} = |n|^2 a_n + \sum_{\Gamma(n)} a_{n_1}\overline{a_{n_2}}a_{n_3}$$
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Resonant Birkhoff
Normal Form

Infinite System of ODE's
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Finite System
of ODE's
(Toy Model System)

Reduction of the problem

PDE (*NLS*)

$$-i\partial_t u + \Delta u = |u|^2 u$$

→ Fourier: $u(t,x) = \sum a_n(t)\mathrm{e}^{\mathrm{i}n\cdot x}$ →

Infinite System of ODE's ($\mathcal{F}NLS$)

$$-\mathrm{i}\frac{\mathrm{d}a_n}{\mathrm{d}t} = |n|^2 a_n + \sum_{\Gamma(n)} a_{n_1}\overline{a_{n_2}}a_{n_3}$$

$$\Gamma(n) = \{n_1 - n_2 + n_3 = n\}$$

↓ Resonant Birkhoff Normal Form ↓

Infinite System of ODE's ($R\mathcal{F}NLS$)

$$-\mathrm{i}\frac{\mathrm{d}r_n}{\mathrm{d}t} = |n|^2 r_n + \sum_{\Gamma_{\mathrm{res}}(n)} r_{n_1}\overline{r_{n_2}}r_{n_3}$$

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← Initial condition in a finite set of modes ←

Finite System of ODE's (Toy Model System)

Reduction of the problem

PDE (NLS)

$$-i\partial_t u + \Delta u = |u|^2 u$$

Fourier

$$u(t,x) = \sum a_n(t) \mathrm{e}^{\mathrm{i} n \cdot x}$$

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Resonant Birkhoff Normal Form

Infinite System of ODE's

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Initial condition in a finite set of modes

Finite System of ODE's (Toy Model System)

Instability

The Toy Model System: Reformulation of main theorem

The Toy Model System is:

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} = -\mathrm{i}|b_j|^2 b_j + 2\mathrm{i}\overline{b_j}\left(b_{j-1}^2 + b_{j+1}^2\right), \quad j = 1, \ldots, N \tag{2}$$

with $b_j \in \mathbb{C}$ and the convention that $b_0(t) = b_{N+1}(t) = 0$.

Theorem 2 (Colliander et al. 10)

Given $N > 1$, $\epsilon \ll 1$, there is initial data $b(0) = (b_1(0), \ldots, b_N(0)) \in \mathbb{C}^N$ for (2) and there is a time $T = T(N, \epsilon)$ so that

$$\begin{aligned} |b_3(0)| \geq 1 - \epsilon, \quad |b_j(0)| \leq \epsilon, \quad j \neq 3 \\ |b_{N-2}(T)| \geq 1 - \epsilon, \quad |b_j(T)| \leq \epsilon, \quad j \neq N-2. \end{aligned}$$

Properties of the Toy Model System

It is a N-d.o.f Hamiltonian system with Hamiltonian:

$$H(b, \bar{b}) = \sum_{j=1}^{N} \left(\frac{1}{4} b_j^2 \bar{b}_j^2 - \frac{1}{2} b_j^2 \bar{b}_{j-1}^2 - \frac{1}{2} \bar{b}_j^2 b_{j-1}^2 \right), \tag{3}$$

that presents an additional conserved quantity, the total mass: $M_N(b_1, \ldots, b_N) = \sum_{j=1}^{N} |b_j|^2$, that will be fixed at 1.

It has N invariant objects (periodic orbits):

$$\mathcal{T}_j := \{(b_1, \ldots, b_N) \in \mathbb{C}^N \, : \, |b_j| = 1; \, b_k = 0 \, \forall k \neq j\}.$$

Theorem 2 $\rightsquigarrow$ Find an orbit which initially is ϵ-close to $\mathcal{T}_3$ and after some time T is ϵ-close to $\mathcal{T}_{N-2}$

Properties of the Toy Model System

How? Take advantage of the existence of an explicit family of heteroclinic connections between these periodic orbits (the 2-d.o.f system is integrable):

$$\gamma^{+}_{j,j+1} : \{\mathcal{T}_j \to \mathcal{T}_{j+1}\} \qquad \gamma^{-}_{j+1,j} : \{\mathcal{T}_{j+1} \to \mathcal{T}_j\}$$

The interesting heteroclinics are the forward ones: $\gamma^{+}_{j,j+1} : \{\mathcal{T}_j \to \mathcal{T}_{j+1}\}$:

$$b_k = 0 \quad \forall k \neq j, j+1$$

$$b_j(t) = \frac{1}{\sqrt{1 + e^{2\sqrt{3}t}}} e^{-i(t+\alpha)} \quad b_{j+1}(t) = \frac{1}{\sqrt{1 + e^{-2\sqrt{3}t}}} e^{-i(t+\alpha)} e^{i\pi/3}$$

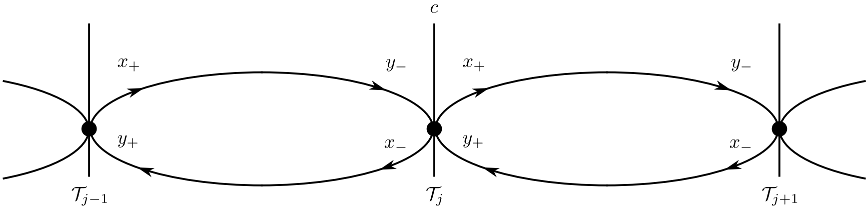
with $\alpha \in \mathbb{T}$.

Linear stability of the periodic orbits $\mathcal{T}_j$

- Local change of coordinates around $\mathcal{T}_j$:

$$b_j = re^{i\theta} \qquad b_k = c_k e^{i\theta} \text{ for } k \neq j.$$

- One can see that the equations for c_k do not depend on θ. The periodic orbit $\mathcal{T}_j$ has collapsed to an equilibrium point.
- The two first neighbors (c_{j-1} and c_{j+1}) split into only 4 real hyperbolic directions, say x_-, y_- and x_+, y_+ (including the heteroclinics) with the same couple of eigenvalues $\pm\sqrt{3}$.
- The rest of modes (the "far modes") are $N-3$ complex elliptic directions, say $c \in \mathbb{C}^{N-3}$.



Summary

Non-transverse diffusion

- The Toy Model System presents a non-transverse heteroclinic chain.
- Non-transverse heteroclinic chain may impede diffusion in some cases.
- One can construct easy examples where non-transverse diffusion is possible.
- An easy theorem is introduced which contains a new methodology for diffusion: losing suitable dimensions.
- It is applicable to a more general setting (including the Toy Model System)

Applying the new scheme to the Toy Model System

- We present a slightly different theorem
- We apply the scheme of losing dimensions in the Toy Model System

Non-transverse intersection in the Toy Model System

- Write the Toy Model System in action-angle coordinates: $H(I, \theta)$.
- Look for a symplectic change of coordinates, (J, φ), that contains the total mass as a coordinate: $J_1 = \sum 2I_k$.
- Its conjugated angle, φ_1, is a cyclic coordinate. We have reduced the number of degrees of freedom by one.

After an additional symplectic change (K, ψ), restrict the 2 d.o.f. Toy Model System in one mode: $H(K_2, \psi_2)$. The equations of motion are:

$$\dot{\varphi}_1 = -1 + 2K_2\,(1 + 2\cos 2\psi_2)$$

$$\dot{K}_2 = 8K_2 \sin 2\psi_2 \left(\frac{1}{2} - K_2\right)$$

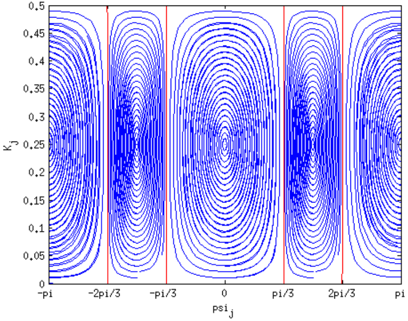
$$\dot{\psi}_2 = 2\,(1 + 2\cos 2\psi_2)\left(\frac{1}{2} - K_2\right) - 2K_2\,(1 + 2\cos 2\psi_2)\,.$$

and the periodic orbits $\mathcal{T}_1$ and $\mathcal{T}_2$ are:

$$\mathcal{T}_1 = \{K_2 = 0, \psi_2 = \pi/3\} \qquad \mathcal{T}_2 = \{K_2 = 1/2, \psi_2 = \pi/3\}$$

The Toy Model System in action-angle coordinates

The phase portrait for this system is:



Notice that if we restrict the system to $\gamma_{1,2}$, we obtain:

$$\dot{\varphi}_1 = -1 \quad \dot{K}_2 = 4\sqrt{3}K_2\left(\frac{1}{2} - K_2\right) \quad \dot{\psi}_2 = 0.$$

The Toy Model System in action-angle coordinates

Using the cyclic coordinate, φ_1, we can define a Poincaré map that is explicit over the heteroclinic

$$P(K_2) = \frac{1}{2 + \frac{1-2K_2}{K_2} \mathrm{e}^{-4\pi\sqrt{3}}}.$$

Its positive (negative) iterates tend to $K_2 = 1/2$ ($K_2 = 0$) for any point K_2 in the segment $\Rightarrow$ each point is heteroclinic.

We have obtained a map, P, for which the invariant manifolds of two consecutive fixed points coincide, that is, it is not transverse.

If we recover the flow, we will have that the heteroclinic connecting the periodic orbits $\mathcal{T}_1$ and $\mathcal{T}_2$ is not transverse.

Transverse vs non-transverse

Consider a two dimensional map with four fixed points, located at the points:

$$p_0 = (0,0) \quad p_1 = (1,0) \quad p_2 = (1,1) \quad p_3 = (2,1).$$

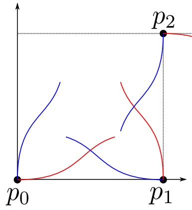


Figure : Transverse situation

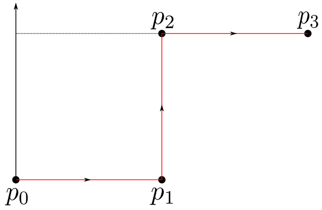


Figure : Non-transverse situation

Can we connect p_0 with p_3 in both cases?

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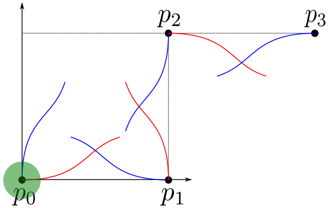

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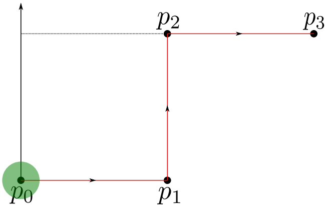

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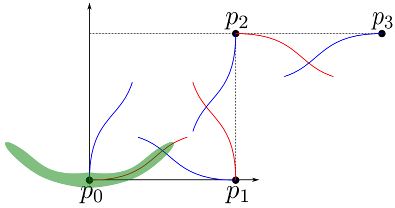

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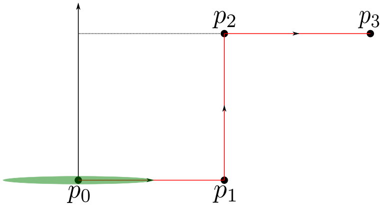

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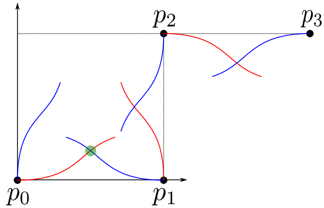

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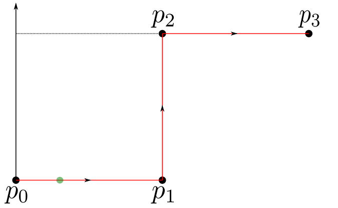

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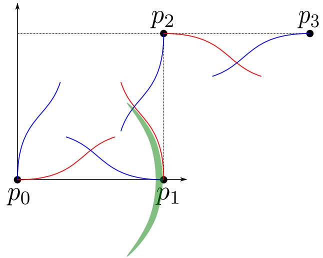

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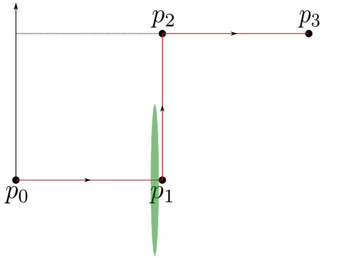

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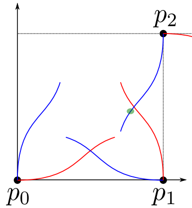

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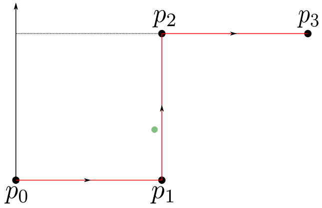

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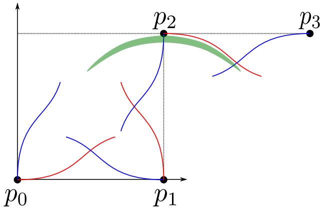

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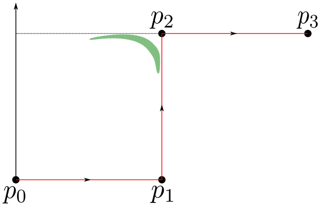


Figure : Non-transverse situation

Can we connect p_0 with p_3 in both cases?

Transverse vs non-transverse

- In the transverse situation there are no geometric obstructions in shadowing the heteroclinic chain.
- In the non-transverse case, we can see that, in general, we cannot visit as many invariant objects as we want.

Why can the authors of [Colliander et al. 10] connect N periodic orbits in the Toy Model System?

The main reason is the large dimension of the system and the fact that each connection takes place in a new direction that has not been used in the past.

Modification of the Toy Model System

Let $H(K,\psi)$ be the reduced Hamiltonian (of $(N-1)$-d.o.f for which the periodic orbits are equilibrium points) of The Toy Model System in action-angle coordinates and add the following terms that only depend on the actions (the equations for the actions remain invariant):

$$\tilde{H}(K,\psi) = H(K,\psi) + K_3 - 2\sum_{j=3}^{N} K_j K_{j-1} + 2\sum_{j=3}^{N-1} K_{j+1} K_{j-1}.$$

A concrete configuration of the angles, $\psi = \pi/3$, is an invariant subspace. The equations for the actions in this subspace are:

$$\left\{ \begin{array}{rcl} \dot{K}_2 & = & 4\sqrt{3}\left(\frac{1}{2} - K_2\right)\left(K_2 - K_3\right) \\ \dot{K}_i & = & 4\sqrt{3}\left(K_{i-1} - K_i\right)\left(K_i - K_{i+1}\right) \text{ for } i \neq 2, N \\ \dot{K}_N & = & 4\sqrt{3}K_N\left(K_{N-1} - K_N\right). \end{array} \right. \tag{4}$$

Modification of the Toy Model System

- It has N equilibrium points:

$$p_1 = (0, \ldots, 0) \qquad p_j = \left(\frac{1}{2}, \overset{j-1}{\ldots}, \frac{1}{2}, 0, \overset{N-j}{\ldots}, 0\right) \quad \text{for } j = 2 \ldots N.$$

- The segments that connect two consecutive equilibrium points,

$$C_{j-1,j} = \left\{ \left(\frac{1}{2}, \overset{j-2}{\ldots}, \frac{1}{2}, K_j, 0, \overset{N-j}{\ldots}, 0\right), \, 0 \leq K_j \leq \frac{1}{2} \right\}$$

  for $j = 2 \ldots N$ are invariant and heteroclinic.

- Notice that each connection takes place in a new direction, not used before.
- We have obtained a simpler example for which we could prove the same scheme of diffusion.
- However, we want to construct an even easier example where we can guarantee the diffusion.

The integrable example

Inspired by the structure of system (4), we impose the previous conditions in a generic system defined by a polynomial of degree two. Simplifying as much as possible, we obtain the following system:

$$\dot{x} = F(x)$$

with

$$\begin{cases} F_1(x) &= \lambda_1 x_1 - \lambda_1 x_1^2 \\ F_i(x) &= (\lambda_i - \mu_i)x_i - \lambda_i x_i^2 + \mu_i x_i x_{i-1} \qquad \text{for } 1 < i \leq n \end{cases} \tag{5}$$

with $\lambda_i > 0$ for $1 \leq i \leq n$ and $\mu_i \in \mathbb{R}$ for $1 < i < n$. Note that it is a triangular system and therefore integrable by quadratures.

The integrable example

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = \frac{1}{10}.$$

$$(0,0,0,0) \to (1,0,0,0) \to (1,1,0,0) \to (1,1,1,0) \to (1,1,1,1)$$

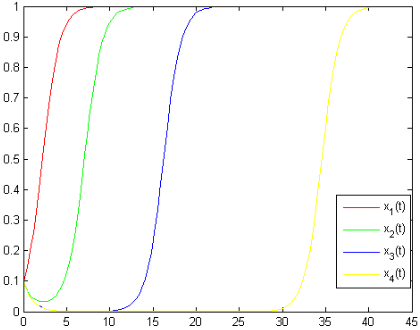


Figure : Solution of system (5) for $\lambda = 1$ and $\mu = 2$.

Summary and remarks

- Toy Model System: Non-transverse heteroclinic.
- The mechanism for diffusion does not rely on the Arnold mechanism.
- If not Arnold diffusion, lack of geometric explanation for diffusion.
- In addition, the lack of transversality can forbid the connection.
- However, we have detected the reason why the connection could be possible in the Toy Model System.
- The geometric mechanism relies on the fact that we are dealing with a high dimensional system and that each new connection is defined by a direction that has not been used before.
- Example integrable by quadratures for which the diffusion takes place.
- The found diffusion has nothing to do with the integrability of the system.

A Theorem for an easy situation

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism with the following properties:

1. The points $p_i = (1, \overset{i}{\ldots}, 1, 0, \overset{n-i}{\ldots}, 0)$ are fixed under f for $i = 0 \ldots n$.
2. The segments C_i that connect the points p_{i-1} and p_i,
$$C_i = \{(1, \overset{i-1}{\ldots}, 1, t, 0, \overset{n-i}{\ldots}, 0), 0 \leq t \leq 1\}$$
for $1 \leq i \leq n$ are invariant under f and, for all $x \in C_i$:
$$\lim_{k\to\infty} f^k(x) = p_i \qquad \lim_{k\to-\infty} f^k(x) = p_{i-1}.$$
3. At each point p_i the i-th direction is the strong stable direction and the $(i+1)$-th is the strong unstable direction. This means:
$$Df(p_i)e_i = \mu_i e_i, \quad |\mu_i| < 1$$
$$Df(p_i)e_{i+1} = \lambda_i e_{i+1}, \quad |\lambda_i| > 1$$
4. The past directions, defined by $\vec{e}_1, \ldots, \vec{e}_{i-1}$, are contracting directions around the fixed point p_i but with a lower rate than μ_i. The future directions, defined by $\vec{e}_{i+2}, \ldots, \vec{e}_n$, are expanding directions around the fixed point p_i but with a lower rate than λ_i.

A Theorem for an easy situation

Theorem 3

Under the previous assumptions, for all $\epsilon > 0$ *there exist a point* x_0 *and a sequence of integers* $0 = k_0 < k_1 < \cdots < k_n$ *such that:*

$$||f^{k_i}(x_0) - p_i|| < \epsilon \quad i = 0, \ldots, n.$$

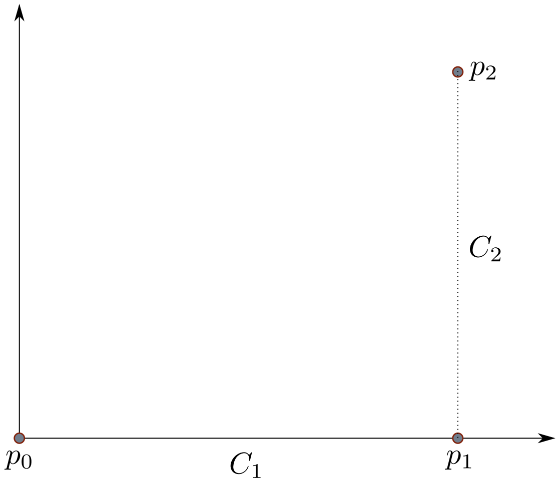
Remark

Notice that we connect $n+1$ given points in a n dimensional space. We cannot guarantee that the result is valid for more points. This Theorem is designed to be applied in high dimensional systems.

Remark

This is a very simple version of the Theorem that can be widely generalized. In particular, to the Toy Model System where one has two dominant stable directions and two dominant stable directions.

Sketch of the proof: losing dimensions



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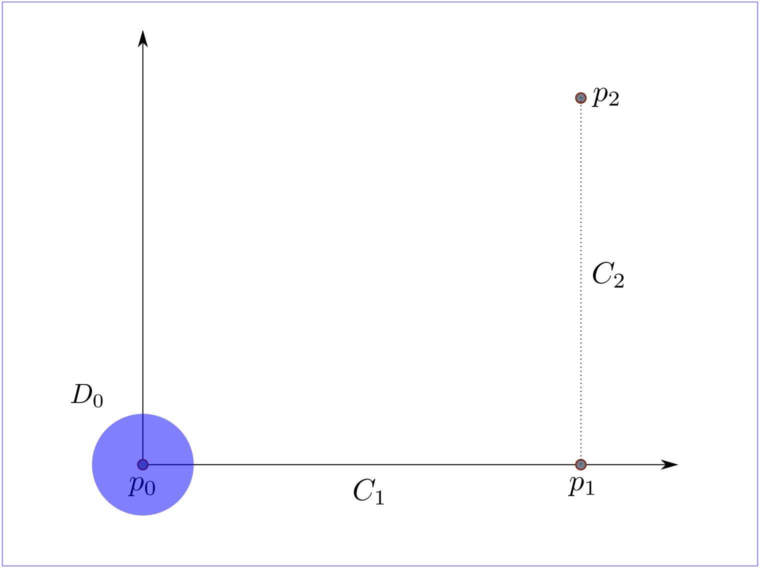

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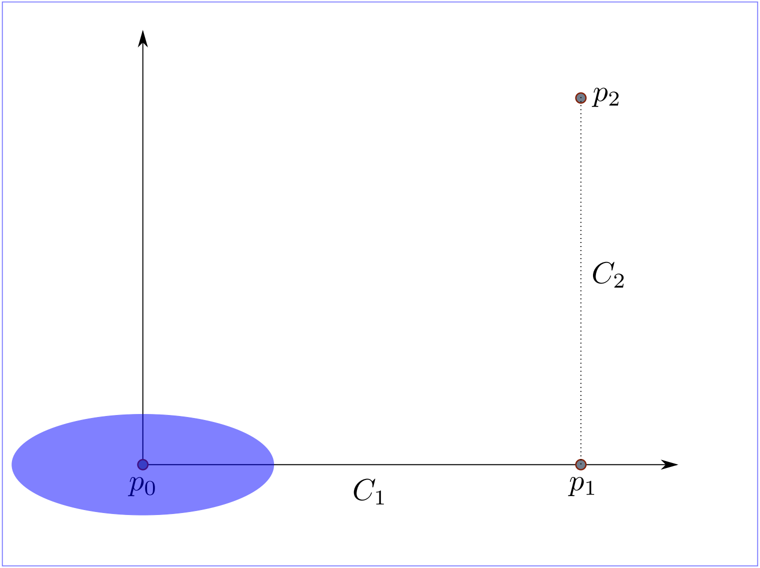

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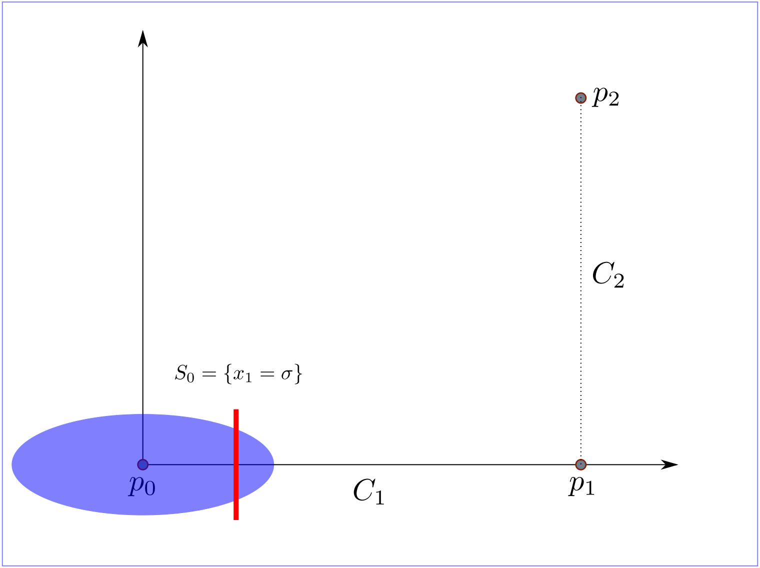

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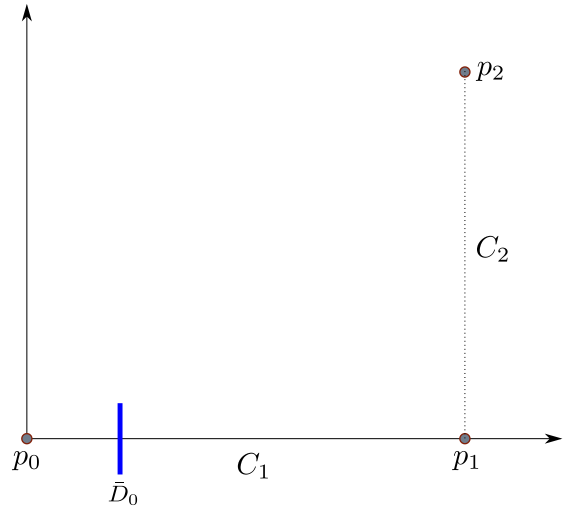

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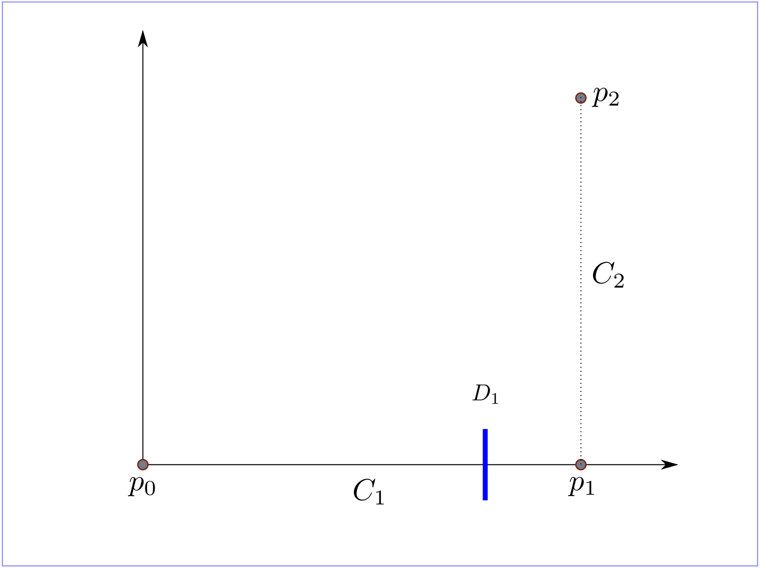

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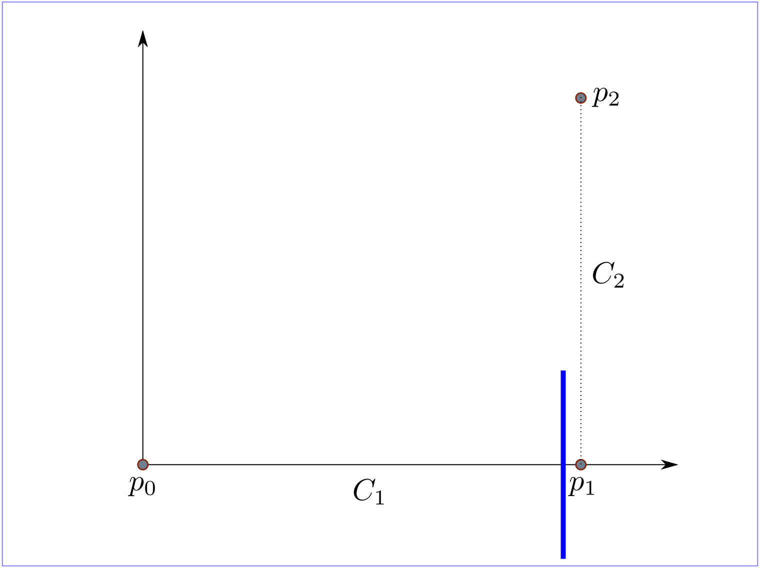

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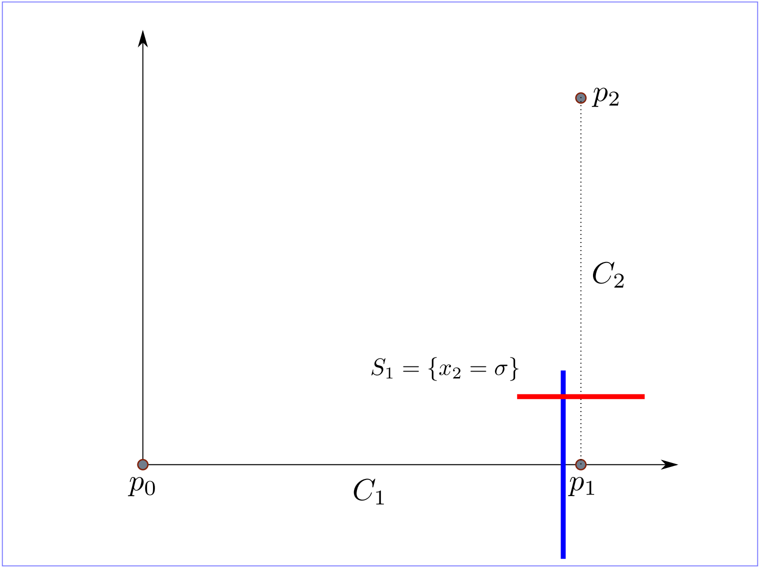
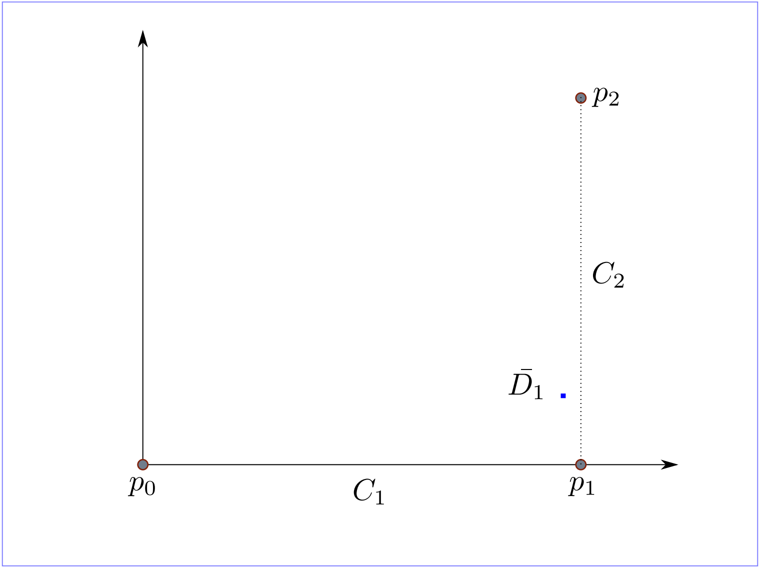


Sketch of the proof: losing dimensions



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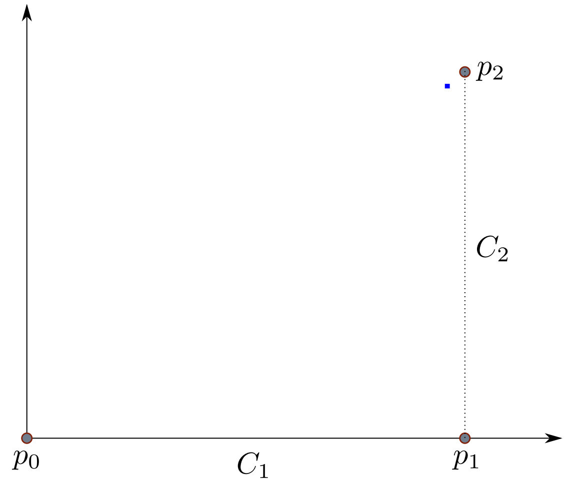


Proof of Theorem 3 using covering relations

- We present a proof for the linearized map around the fixed points.
- The proof is based in the language of *h*-sets and covering relations.
- We define a sequence of h-sets with a decreasing number of exit directions.
- We prove that the sequence of *h*-sets are related with covering relations

Generalizing Theorem 3

Goal: To produce a Theorem just like Theorem 3 for a wider class of system that would include the Toy Model System.

This is a very idealistic goal

Alternative: To prepare a general Theorem and show that its proof is equivalent to showing a chain of covering relations between concrete h-sets.

- Description of the class of systems: allowing several dominant directions around each equilibrium point.
- Construction of h-sets with the structure of decreasing number of exit directions.
- Purely topological shadowing Theorem: Assuming that some covering conditions hold, we prove a shadowing Theorem.

Diffusion in The Toy Model System

Goal: To use the ideas and tools obtained in the previous one to prove Theorem 4.
Consequence: We would have identified the diffusion mechanism that appears in the Toy Model System.

Theorem 4

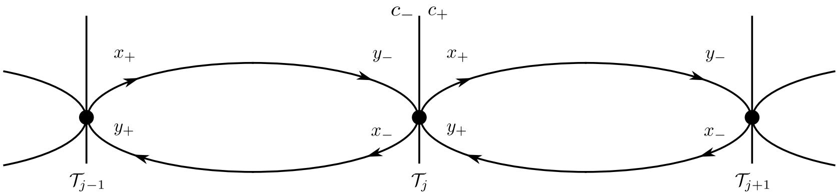
Let $N > 1$, $\delta \ll 1$. There exists a time, $T_0^* = T_0^*(N, \delta)$ such that, for all time $T^* \geq T_0^*$, there exists an initial data $b(0) = (b_1(0), \ldots, b_N(0)) \in \mathbb{C}^N$ for (2) such that

$$|b_3(0)| \geq 1 - \delta, \quad |b_j(0)| \leq \delta, \quad j \neq 3$$
$$|b_{N-2}(T^*)| \geq 1 - \delta, \quad |b_j(T^*)| \leq \delta, \quad j \neq N-2.$$

Instead of working with *h*-sets, we are going to present a different but equivalent argument with disks.

Sketch of the argument

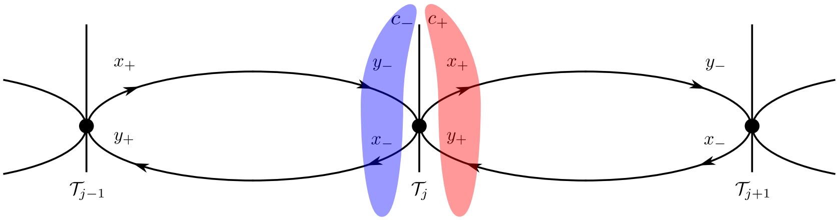
Recall its structure:



- The difficulty here is the existence of four hyperbolic directions (instead of two) with the same expansion rate. We must lose all the four hyperbolic directions.

Sketch of the argument

Recall its structure:



- The difficulty here is the existence of four hyperbolic directions (instead of two) with the same expansion rate. We must lose all the four hyperbolic directions.
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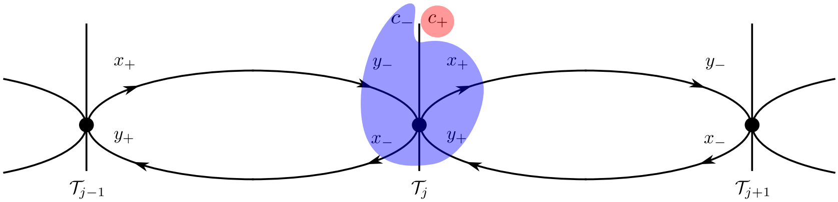
Sketch of the argument

Recall its structure:



- The difficulty here is the existence of four hyperbolic directions (instead of two) with the same expansion rate. We must lose all the four hyperbolic directions.
- We assume the blue coordinates already fixed at this step. The red are the still free ones.
- We intersect our disk with a codimension 2 manifold located in the desired direction, losing 2 directions.

The disks

The way that we illustrate such a splitting of fixed and free coordinates is through what we call a generic disk.

Definition 1

Let $z = (p, f)$. A *generic disk* is:

$$D^{\mathrm{gen}} = \{z = (p, f) \,:\, p = m(f), \, |f| \le r_f\},$$

where m is a map from $B_{r_f}(0) \subset \mathbb{R}^{n_f}$ to $\mathbb{R}^{n_p}$ such that

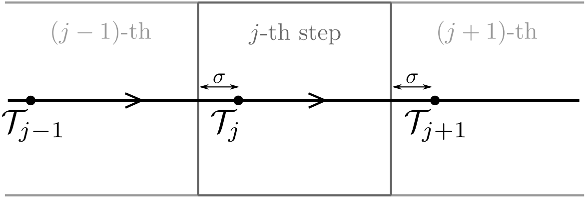
$$||m|| = \sup_{f \in B_{r_f}(0)} ||m(f)|| \le r_p.$$

We say that f is free and p is fixed.

With this definition we can understand better what we mean by fixed (past) or free (future) coordinate.

Steps

We will split the shadowing argument in $N-5$ steps, one for each connection between two consecutive heteroclinics. So, for $j \in \{3, \ldots, N-2\}$, we will have this schematic situation:



The junction point between steps is determined by some distance, that we call macroscopic, σ, to the periodic orbit. This is a small global parameter that does not depend on the step.

The Incoming Disk

Definition 2

An *incoming disk* is defined as

$$D_j = \Big\{ z \in \mathcal{S}_N^j \, : c_- = m_0^{c_-}(x_+, y_+, c_+),\ x_- = m_0^{x_-}(x_+, y_+, c_+),$$
$$y_- = \sigma + m_0^{y_-}(x_+, y_+, c_+),$$
$$|x_+| \leq r_{x_+}^0,\ |y_+| \leq r_{y_+}^0,\ |c_{+,k}| \leq r_{c_{+,k}}^0 \,\Big\},$$

where we assume

$$\left| m_0^{c_{-,k}} \right| \leq r_{c_{-,k}}^0 \quad \left| m_0^{x_-} \right| \leq r_{x_-}^0 \quad \left| m_0^{y_-} \right| \leq r_{y_-}^0 .$$

We assume

$$r^0 = \max\left\{ r_{c_-}^0, r_{x_-}^0, r_{y_-}^0, r_{x_+}^0, r_{y_+}^0, r_{c_+}^0 \right\}$$

is bounded by some microscopic quantity, much smaller than σ

The singular and regular problem

We distinguish two different regimes:

Singular problem:

- Flow close to a partially hyperbolic partially elliptic equilibrium point.
- Losing dimensions.
- From the incoming disk to the outgoing disk

Regular problem:

- Flow close to the heteroclinic and change of coordinates.
- From the outgoing disk to the new incoming disk.

The Singular Problem: Options for losing the dimensions

We have two possibilities:

- Compute the evolution of the whole incoming disk and intersect it with a codimension-two section located in the desired place. Since the outgoing heteroclinic is defined by the x_+-axis we define the section as:

$$S_j = \{x_+ = \sigma + x_+^*,\, y_+ = y_+^*\}. \tag{6}$$

- Use a Shilnikov Problem that computes only the evolution of the part of our disk that ends in the desired section S_j.

We will use the Shilnikov approach.

The Shilnikov Problem

These kind of problems were introduced in [Shilnikov 67].

Definition 3

Let F be a vector field in $\mathbb{R}^n$. Consider the following system of ordinary differential equations:

$$\dot{z} = F(z). \tag{7}$$

Split the coordinates in $z = (z_1, z_2)$ and $n_1 + n_2 = n$. Let $(\tau; z_1^0, z_2^1)$ be the *Shilnikov data*, where $\tau \in \mathbb{R}^+$, $z_1 \in \mathbb{R}^{n_1}$ and $z_2 \in \mathbb{R}^{n_2}$.
$z(t)$ is a solution for the *Shilnikov problem* with data $(\tau; z_1^0, z_2^1)$ if:

$$\dot{z}(t) = F(z(t)) \text{ for } t \in [0, \tau] \qquad z_1(0) = z_1^0 \qquad z_2(\tau) = z_2^1.$$

- Note that, for $\tau = 0$ we recover the initial value problem.
- The existence of solution is not guaranteed. A proof for existence for saddle equilibrium points for an adequate splitting for the coordinates is given in [Gonchenko-Shilnikov-Turaev 92],[Deng 89].

The Shilnikov Problem: application to our case

- We are going to set all the past coordinates (c_-, x_-, y_-) for time $t = 0$ in the incoming disk.
- Fix the future hyperbolic coordinates (x_+, y_+) at time $t = \tau$ at the section, S_j, described above.
- We want the rest of free coordinates, c_+, to remain free after this step. We are going to consider a Shilnikov problem for each admissible value of c_+ in the incoming disk at time $t = 0$.
- The family of solutions will form a disk of two dimensions less located in the outgoing section. We will call to this disk, the outgoing disk.

The Shilnikov Problem: remarks on the modification

- We are generalizing the Shilnikov problem to a non hyperbolic equilibrium point, that is the case considered in [Shilnikov 67], [Gonchenko-Shilnikov-Turaev 92] and [Deng 89].

- We are not using the standard partition in initial and final conditions. In [Shilnikov 67], [Gonchenko-Shilnikov-Turaev 92] and [Deng 89] stable at $t = 0$ and unstable at $t = \tau$.

- We always keep the linear part of the solution as a reference. We obtain bounds of the deviation of the true solution with respect to the linear one.

The heteroclinic channel

The heteroclinic connection is defined in the x_+-axis with the explicit expression:

$$x_+^h(t) = \frac{1}{\sqrt{1 + \frac{1-\sigma^2}{\sigma^2} \mathrm{e}^{-2\sqrt{3}t}}},$$

and zero all the other components.
This solution starts at the macroscopic distance σ for $t = 0$ and, after a time

$$T = \frac{1}{\sqrt{3}} \ln\left(\frac{1-\sigma^2}{\sigma^2}\right), \tag{8}$$

it ends up at a point $x_+^h(T) = \sqrt{1-\sigma^2}$, that is, at a macroscopic distance of the following equilibrium point.
Using crude Gronwall estimates we can prove that the outgoing disk flows close to the heteroclinic and reaches the proximity of the next equilibrium point.

The change of coordinates, from $\mathcal{T}_j$ to $\mathcal{T}_{j+1}$

To end the step we have to change the coordinates to the ones that refer the motion to the periodic orbit $\mathcal{T}_{j+1}$.

$$\underbrace{c_1, \ldots, c_{j-2}, x_-, y_-,}_{\tilde{c}_1, \ldots, \tilde{c}_{j-1},} \quad \underbrace{x_+, y_+,}_{\tilde{x}_-, \tilde{y}_-,} \quad \underbrace{c_{j+2},}_{\tilde{x}_+, \tilde{y}_+,} \quad \underbrace{c_{j+3} \ldots, c_N}_{\tilde{c}_{j+3} \ldots, \tilde{c}_N}$$

We see that the change of coordinates of the evolved outgoing disk contains the new incoming disk D_{j+1}.

Comments on the proof

- The validity of this geometric inductive argument depends on the right choices of the sizes of our disks.
- We always take the linear part of the system to guide our procedure and to check if the argument is feasible at least with it. However, when we consider the nonlinear terms we have to deal with:
  - Not so uncoupled system: Need to straighten the invariant manifolds in the singular problem.
  - Presence of resonant terms makes the linear part not to approximate a true solution at all scales: Need to modify the sizes predicted by the linear part.
- To guarantee the whole transition we do not allow any relation between σ and τ. Then, we need to perform a quasi-normal form change of coordinates.

Differences with previous works concerning the Singular problem

- Colliander et al. 10 Do not need to straighten invariant manifolds

- Guardia-Kaloshin 15
  - Want to obtain an estimate for the diffusion time, so they do not want to flow as close to the periodic orbits as we do.
  - They use the resonant system instead of the linear part as a reference.
  - Thus, they perform a change into normal form (that includes the straightening of invariant manifolds and our quasi-normal form change).

Final comments, future work and possible open questions:

- Our Theorem holds for any time T^* large enough.
  - Increasing the time, we can obtain a different solution that flows closer to the heteroclinic chain, improving, then, Theorem 2.
  - This unboundedness of T^* forced us to obtain sharper bounds in the whole discussion and to perform a quasi-normal form change.
  - If we allowed an upper bound for the time T^* (that would depend on σ) we could possibly work without such precise tools.
- We could think wether the shadowing of an infinite sequence of periodic orbits is feasible. That does not mean unbounded growth of the Sobolev norm in (1) but it will have its own particular interest.
- Think of other examples where the mechanism can be applied.

Thanks for your attention!

The Shilnikov Theorem

Consider a system of the form:

$$\left\{\begin{array}{lcl} \dot{\xi}_\pm & = & \sqrt{3}\xi_\pm + R^{\xi_\pm}(\xi,\eta,c) \\ \dot{\eta}_\pm & = & -\sqrt{3}\eta_\pm + R^{\eta_\pm}(\xi,\eta,c) \\ \dot{c}_k & = & ic_k + R^{c_k}(\xi,\eta,c) \end{array}\right. \quad (9)$$

where the nonlinearities have the following expression:

$$\left\{\begin{array}{lcl} R^{\xi_\pm}(\xi,\eta,c) & = & \xi_\pm R^{\xi_\pm}_{\mathrm{unc}}(\xi,\eta,c) + \xi_\mp\eta_\pm R^{\xi_\pm}_{\mathrm{coup}}(\xi_\mp,\eta,c) \\ R^{\eta_\pm}(\xi,\eta,c) & = & \eta_\pm R^{\eta_\pm}_{\mathrm{unc}}(\xi,\eta,c) + \eta_\mp\xi_\pm R^{\eta_\pm}_{\mathrm{coup}}(\xi,\eta_\mp,c) \\ R^{c_k}(\xi,\eta,c) & = & c_k R^{c_k}_1(\xi,\eta,c) + \bar{c}_k R^{c_k}_2(\xi,\eta,c) \end{array}\right. \quad (10)$$

The Shilnikov Theorem

Let $\gamma_0 > 0$ and such that:

$$\gamma_0 \leq \frac{2\sqrt{3}}{3K},$$
$$\frac{14K}{\sqrt{3}}(1+\gamma_0)^2\gamma_0^2 \leq \frac{1}{3}.$$

Let $1/2 < \epsilon < 1$, $0 < \gamma \leq \gamma_0$, $k \geq 0$ and $\tau \in \mathbb{R}$ large enough.
Take $\xi_{-,0}, \eta_{-,0}, \xi_{+,1}, \eta_{+,1} \in \mathbb{R}$ and
$\zeta = (\zeta_1, \ldots, \zeta_{j-2}, \zeta_{j+2}, \ldots, \zeta_N) \in \mathbb{C}^{N-3}$, such that:

$$|\xi_{-,0}| = \gamma_{\xi_-} \leq \frac{1}{2\sqrt{3}}\gamma\tau^k e^{-2\sqrt{3}\tau} \qquad |\eta_{-,0}| = \gamma_{\eta_-} \leq \frac{1}{2\sqrt{3}}\gamma$$
$$|\xi_{+,1}| = \gamma_{\xi_+} \leq \frac{1}{2\sqrt{3}}\gamma \qquad |\eta_{+,1}| = \gamma_{\eta_+} \leq \frac{1}{2\sqrt{3}}\gamma\tau^k e^{-2\sqrt{3}\tau}$$
$$|\zeta_k| = \gamma_k \leq \frac{1}{N-3}\frac{1}{2\sqrt{3}}\gamma\tau^k e^{-\epsilon\sqrt{3}\tau}$$

The Shilnikov Theorem

Then there exists a unique solution defined for $t \in [0, \tau]$ such that:

$$\begin{array}{rcll} \xi_-(0) & = & \xi_{-,0} & \eta_-(0) = \eta_{-,0} \\ \xi_+(\tau) & = & \xi_{+,1} & \eta_+(\tau) = \eta_{+,1} \\ c_k(0) & = & \zeta_k & \end{array}$$

In addition, the solution deviates from the linear solution as:

$$\left|\xi_-(t) - \xi_{-,0} e^{\sqrt{3}t}\right| \leq \frac{1}{2}\gamma \left(\gamma_{\xi_-} + \gamma_{\eta_-} \tau e^{-2\sqrt{3}\tau}\right) e^{\sqrt{3}t} \tag{11}$$

$$\left|\eta_-(t) - \eta_{-,0} e^{-\sqrt{3}t}\right| \leq \frac{1}{2}\gamma \left(\gamma_{\eta_-} + \frac{1}{2}\gamma_{\xi_-} e^{\sqrt{3}\tau}\right) e^{-\sqrt{3}t} \tag{12}$$

$$\left|\xi_+(t) - \xi_{+,1} e^{\sqrt{3}(t-\tau)}\right| \leq \frac{1}{2}\gamma \left(\gamma_{\xi_+} + \frac{1}{2}\gamma_{\eta_+} e^{\sqrt{3}\tau}\right) e^{\sqrt{3}(t-\tau)} \tag{13}$$

$$\left|\eta_+(t) - \eta_{+,1} e^{-\sqrt{3}(t-\tau)}\right| \leq \frac{1}{2}\gamma \left(\gamma_{\eta_+} + \gamma_{\xi_+} \tau e^{-2\sqrt{3}\tau}\right) e^{-\sqrt{3}(t-\tau)} \tag{14}$$

$$\left|c_k(t) - \zeta_k e^{it}\right| \leq \frac{1}{2}\gamma\gamma_k \tag{15}$$

[Bourgain 96, Staffilani 97] proved at most polynomial growth of Sobolev norms if one replaces the nonlinearity with a more general polynomial.