Concept of Stability as a Whole of a Family of Fibers Maps for C^1-Smooth Skew Products and Its Generalization

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Preliminaries I

Let $I = I_1 \times I_2$ be a closed rectangle in the plane (I_1, I_2 are closed intervals). We consider a skew product of maps of an interval, i. e. a dynamical system $F : I \to I$, where

$$F(x, y) = (f(x), g_x(y)), \quad \text{and} \quad g_x(y) = g(x, y), \ (x; y) \in I. \quad (1)$$

By formula (1) the equality

$$F^n(x, y) = (f^n(x), g_{x,n}(y)), \quad \text{where} \quad g_{x,n} = g_{f^{n-1}(x)} \circ \ldots \circ g_x, \quad (2)$$

is valid for every natural number n and every point $(x; y) \in I$.
Let, as usually, $T^0(I)$ ($T^1(I)$) be the space of all continuous (all C^1-smooth) skew products of maps of an interval with the standard C^0-norm (the standard C^1-norm).

Preliminaries II

Denote by $C^1_{\partial_k}(I_k)$ ($k = 1, 2$) the subspace of the space $C^1(I_k)$ (of C^1-smooth maps of the segment I_k into itself with the standard C^1-norm), which consists of all maps $\psi \in C^1(I_k)$ satisfying the condition of ψ-invariance of the boundary ∂I_k of the segment I_k:

$$\psi(\partial I_k) \subseteq \partial I_k.$$

A map $\xi \in C^1_{\partial_k}(I_k)$ ($k = 1, 2$) is Ω-stable in C^1-norm (i. e. in the space $C^1_{\partial_k}(I_k)$) if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for every map $\varphi \in B^1_{k,\varepsilon}(\xi)$ one can find δ-closed in C^0-norm to the identity map homeomorphism $h : \Omega(\xi) \to \Omega(\varphi)$ satisfying the equality

$$h \circ \xi|_{\Omega(\xi)} = \varphi|_{\Omega(\varphi)} \circ h, \qquad (3)$$

where $B^1_{k,\varepsilon}(\cdot)$ is ε-neighborhood of a map in the space $C^1_{\partial_k}(I_k)$ (with respect to C^1-norm); $\Omega(\cdot)$ is the nonwandering set of a map.

Properties of C^1-smooth Ω-stable maps of an interval

Proposition 1. *Let* $f \in C^1_\omega(I_1)$. *Then*

(1.1) *either* f *is a map of type* $\prec 2^\infty$ *(i. e. the set of the (least) periods of* f*-periodic points coincides with the set* $\{1, 2, \dots, 2^\mu\}$ *for some* $0 \leqslant \mu < +\infty$*)), and in this case the nonwandering set* $\Omega(f)$ *is finite and consists of hyperbolic periodic points;*

(1.2) *or* f *is a map of type* $\succ 2^\infty$ *(i. e. there exists an* f*-periodic point* x *(*$x \in Per(f)$*) with the (least) period* $n(x) \notin \{2^i\}_{i \geqslant 0}$*), and in this case the nonwandering set* $\Omega(f)$ *is the union of finitely many hyperbolic periodic points and finitely many locally maximal quasiminimal sets, which are hyperbolic, perfect and nowhere dense. ("Locally maximal"means "maximal in a neighborhood of itself".)*

The set $C^1_\omega(I_1)$ *is open and everywhere dense in* $C^1_{\partial_1}(I_1)$.

Choice of the space

Define the space

$$T^1_*(I)$$

of C^1-smooth skew products of maps of an interval (with the standard C^1-norm) as the subspace of the space

$$T^1(I),$$

which consists of skew products of maps of an interval with quotients from the space

$$C^1_\omega(I_1).$$

As it follows from Proposition 1, the set $T^1_*(I)$ is open and everywhere dense in the subspace of the space $T^1(I)$ consisting of skew products with quotient maps from $C^1_{\partial_1}(I_1)$.

Special multifunctions: the Ω-function

Definition 1. *The* Ω*-function of a map* $F \in T^0(I)$ is the multifunction $\zeta^F : \Omega(f) \to 2^{I_2}$ satisfying the equality

$$\zeta^F(x) = (\Omega(F))(x)$$

for any $x \in \Omega(f)$, where $(\Omega(F))(x) = \{y \in I_2 : (x; y) \in \Omega(F)\}$ is the slice of the nonwandering set $\Omega(F)$ by the vertical fiber over a point x, 2^{I_2} is the topological space of closed subsets of I_2 with the exponential topology.
Let n be a natural number,

$$F_n(x,\, y) = (id(x),\, g_{x,\,n}(y)), \quad F_{n,1}(x,\, y) = (f^n(x),\, id(y)),$$

where $id(x)$ and $id(y)$ are the identity maps of the closed intervals I_1 and I_2, respectively, and F_n, $F_{n,\,1} : I \to I$. Then we have:

$$F^n = F_{n,\,1} \circ F_n. \tag{4}$$

Special multifunctions: auxiliary and suitable functions to the Ω-function

Definition 2. *An auxiliary multifunction of a map* $F \in T^1_*(I)$ is a multifunction $\eta_n : \Omega(f) \to 2^{I_2}$ satisfying

$$\eta_n(x) = \Omega(g_{x,\,n})$$

for any $x \in \Omega(f)$, where $\Omega(g_{x,\,n})$ is the nonwandering set of a map $g_{x,\,n} : I_2 \to I_2$.
A function $\overline{\eta}_n : \Omega(f) \to 2^{I_2}$ is said to be *a multifunction suitable to the Ω-function of a map $F \in T^1_*(I)$* if the graph of $\overline{\eta}_n$ in I is the closure of the graph of the auxiliary function η_n. We have

$$\overline{\eta}_n(x) = (\overline{\eta}_n)(x) \quad \text{for any} \quad x \in \Omega(f).$$

Here $(\overline{\eta}_n)(x)$ denotes the slice of the graph of $\overline{\eta}_n$ by the fibre over x (or, equivalently, the slice of the closure of the graph of η_n).

Special multifunctions: approximating functions to the Ω-function

Having defined auxiliary functions η_n (suitable functions $\overline{\eta}_n$) for all $n > 1$, we must move each point $(x; y)$ on the graph of η_n (on the graph of $\overline{\eta}_n$, respectively) to the point $(f^n(x); y)$ using the direct product $F_{n,1}$ (see equality (4)). We can therefore define multifunctions $\eta_{n,1} : \Omega(f) \to 2^{I_2}$ ($\overline{\eta}_{n,1} : \Omega(f) \to 2^{I_2}$), $n > 1$, by the equalities

$$\eta_{n,1}(x) = (F_{n,1}(\eta_n))(x) \quad (\overline{\eta}_{n,1}(x) = (F_{n,1}(\overline{\eta}_n))(x)$$

for any $x \in \Omega(f)$. Here η_n ($\overline{\eta}_n$) is the graph of the corresponding multifunction in I, and $(F_{n,1}(\eta_n))(x)$ ($(F_{n,1}(\overline{\eta}_n))(x)$) is the slice of the set $F_{n,1}(\eta_n)$ (of the set $F_{n,1}(\overline{\eta}_n)$) by the fibre over $x \in \Omega(f)$.

Preliminaries for Decomposition theorem

Let $F \in T^1_*(I)$ be a skew product with a quotient map of type $\succ 2^\infty$. Then by Proposition 1 the perfect part of the nonwandering set of its quotient map $\Omega_p(f)$ is not empty. Let $K(f) \subset \Omega_p(f)$ be a locally maximal quasiminimal set of f, and $\tau(f_{|K(f)})$ be the set of the (least) periods of periodic points of $f_{|K(f)}$. There exist natural numbers $m_0 = m_0(K(f))$, $i_0 = i_0(K(f))$ and a finite subset $N_* = N_*(K(f))$ of the set $\mathbf{N}$ of natural numbers (possibly, empty) such that

$$\tau(f_{|K(f)}) = \{m_0 i\}_{i \geqslant i_0} \cup N_*.$$

We need the following natural numbers:

$$\begin{aligned} m_* &= \underset{K(f) \subset \Omega_p(f)}{\text{l.c.m.}} \{m_0(K(f))\}, \\ n_* &= \underset{K(f) \subset \Omega_p(f)}{\text{l.c.m.}} \{n \in N_*(K(f))\}, \\ i_* &= \max_{K(f) \subset \Omega_p(f)} \{i_0(K(f))\}. \end{aligned}$$

Skew products satisfying condition **H** (strong condition **H**)

Definition 3. We say that a skew product $F \in T^1_*(I)$ with a quotient map of type $\succ 2^\infty$ *satisfies condition* **H** *(strong condition* **H***)* if for any sequence of natural numbers $\{l^*_i\}_{i \geqslant i_*}$ with

$$l^*_i = m_* n_* i, \tag{5}$$

the multifunctions $\overline{\eta}_{l^*_i}$ (the multifunctions $\eta_{l^*_i}$ respectively) are continuous for all $i \geqslant i^*$, where $i^* \geqslant i_*$.

Let us also note that if a skew product $F \in T^1_*(I)$ with a quotient map of type $\succ 2^\infty$ satisfies condition **H** (strong condition **H**) then the sequence $\{\overline{\eta}_{l^*_i}\}_{i \geqslant 1}$ (the sequence $\{\eta_{l^*_i}\}_{i \geqslant 1}$) can only contain a finitely many discontinuous functions.

Main subspaces of the space $T^1_*(I)$

We denote the subspace of $T^1_*(I)$ consisting of skew products whose quotient maps have type $\succ 2^\infty$ and satisfy strong condition **H** by $T^1_{*,1}(I)$,
and the subspace of $T^1_*(I)$ consisting of skew products whose quotients have type $\succ 2^\infty$ and satisfy condition **H**, but do not satisfy strong condition **H**, by $T^1_{*,2}(I)$.
We let $T^1_{*,3}(I)$ denote the subspace of $T^1_*(I)$ consisting of maps with quotients of type $\succ 2^\infty$, each of which has a sequence of suitable functions $\{\overline{\eta}_{I_i}\}_{i>0}$ containing infinitely many discontinuous functions and has a continuous Ω-function.
Finally, we let $T^1_{*,4}(I)$ denote the subspace of $T^1_*(I)$ consisting of maps with quotients of type $\succ 2^\infty$, each of which has a sequence of suitable functions $\{\overline{\eta}_{I_i}\}_{i>0}$ containing infinitely many discontinuous functions and has a discontinuous Ω-function.
The subspaces $T^1_{*,1}(I)$ – $T^1_{*,4}(I)$ are pairwise disjoint.

Decomposition theorem. *Each of the subspaces* T^1_{*i} *for* $1 \leqslant i \leqslant 4$ *is nonempty, and their union* $\bigcup_{i=1}^{4} T^1_{*,\,i}(I)$ *coincides with the part of the space* $T^1_*(I)$ *consisting of skew products with quotients of type* $\succ 2^\infty$*.*
Let $\widetilde{T}^1 * (I)$ be the subspace of maps $F \in T^1_*(I)$ satisfying the inclusion

$$F(\partial I) \subseteq \partial I,$$

where ∂I is the boundary of I.
L.S. Efremova, A decomposition theorem for the space of C^1-smooth skew products with complicated dynamics of the quotient map, Mat.Sb., [English translation], Sb. Math., **204**:11 (2013), pp. 1598пїS1623.

The concept of stability as a whole of a family of fibers maps

Definition 4. We say that *a family of fibers maps of a skew product* $F \in \widetilde{T}^1_*(I)$ *with a quotient map of type* $\succ 2^\infty$ *is stable as a whole in* C^1*-norm* if for any $\delta > 0$ there is a neighborhood $B^1_\varepsilon(F)$ of F in $\widetilde{T}^1_*(I)$ such that for any map $\Phi \in B^1_\varepsilon(F)$, where $\Phi(x,\, y) = (\varphi(x),\, \psi_x(y))$, and any I^*_i ($i \geqslant i^*$ for some $i^* \geqslant i_*$) one can find δ-close to the identity map in C^0-norm homeomorphism

$$H^{<I^*_i>} : \overline{\eta}^F_{I^*_i} \to \overline{\eta}^\Phi_{I^*_i} \qquad \left(H^{<I^*_i>}(x,\, y) = (h_1(x),\, h_{2,\,x}^{<I^*_i>}(y))\right)$$

satisfying the equality

$$h_{2,\,x}^{<I^*_i>}{}_{|\overline{\eta}^F_{I^*_i}(x)} \circ g_{x,\,I^*_i}{}_{|\overline{\eta}^F_{I^*_i}(x)}(y) = \psi_{h_1(x),\,I^*_i}{}_{|\overline{\eta}^\Phi_{I^*_i}(h_1(x))} \circ h_{2,\,x}^{<I^*_i>}{}_{|\overline{\eta}^F_{I^*_i}(x)}(y),$$

where $(x;\, y)$ is a point of the graph of a function $\overline{\eta}^F_{I^*_i}$ in I.

Criterion of stability as a whole of a family of fibers maps

Theorem 1. *A family of fibers maps of a skew product* $F \in \widetilde{T}^1_*(I)$ *with a quotient map of type* $\succ 2^\infty$ *is stable as a whole in* C^1*-norm iff for any* $\delta > 0$ *there exists a neighborhood* $B^1_\varepsilon(F)$ *of* F *in* $\widetilde{T}^1_*(I)$ *such that for any map* $\Phi \in B^1_\varepsilon(F)$ *and any* I^*_i *(*$i \geqslant i^*$ *for* $i^* \geqslant i_*$*) one can find* δ*-close to the identity map in* C^0*-norm homeomorphism* $H^{<I^*_i>} : \overline{\eta}^F_{I^*_i} \to \overline{\eta}^\Phi_{I^*_i}$ *of skew products class possessing the property:*
maps $F_{I^*_i}{}_{|\Omega(f)\times I_2}$ *and* $\Phi_{I^*_i}{}_{|\Omega(\varphi)\times I_2}$ *are* Ω*-conjugate under homeomorphism* $H^{<I^*_i>}$*.*

Qualitative necessary condition of stability as a whole of a family of fibers maps

Theorem 2. *Let $F \in \widetilde{T}^1_*(I)$ have a quotient map of type $\succ 2^\infty$ and have a stable as a whole in C^1-norm family of fibers maps. Then F satisfies strong condition* **H** *and has a continuous Ω-function.*

Theorem 2 shows that C^1-smooth skew products of maps of an interval with stable as a whole family of fibers maps belong to the subspace $T^1_{*,1}(I)$.

Ω-stability in C^1-norm

Definition 5. We say that a map $F \in \widetilde{T}^1_*(I)$ is *Ω-stable in C^1-norm (in the space $\widetilde{T}^1_*(I)$)* if for any $\delta > 0$ there exists $\varepsilon > 0$ such that for any map $\Phi \in B^1_\varepsilon(F)$, $\Phi(x, y) = (\varphi(x), \psi_x(y))$, one can find δ-close in C^0-norm to the identity map homeomorphism

$$H : \Omega(F) \to \Omega(\Phi), \;\; H(x, y) = (h_1(x), h_{2,x}(y)),$$

such that for every point $(x; y) \in \Omega(F)$ the equalities hold:

$$h_1 \circ f_{|\Omega(f)}(x) = \varphi \circ h_{1|\Omega(f)}(x);$$

$$h_{2,f(x)|(\Omega(F))(f(x))} \circ g_{x|(\Omega(F))(x)}(y) =$$

$$\psi_{h_1(x)|(\Omega(\Phi))(h_1(x))} \circ h_{2,x|(\Omega(F))(x)}(y).$$

Criterion of Ω-stability in C^1-norm and its main corollaries

Theorem 3. *A skew product $F \in \widetilde{T}^1_*(I)$ with a quotient map of type $\succ 2^\infty$ is Ω-stable in C^1-norm iff its family of fibers maps is stable as a whole in C^1-norm.*

Corollary 1. *Let a skew product $F \in \widetilde{T}^1_*(I)$ with a quotient map of type $\succ 2^\infty$ be Ω-stable in C^1-norm. Then $F \in \widetilde{T}^1_{*,1}(I)$, and the Ω-function of F is continuous.*
Moreover, for any different periodic points x', x'' ($m(x')$ and $m(x'')$ are the (least) periods of x' and x'' respectively) of every locally maximal quasiminimal set $K(f)$ of a quotient map f, fibers maps $\widetilde{g}_{x'}^{m'/m(x')}$ and $\widetilde{g}_{x''}^{m'/m(x'')}$ are Ω-conjugate, where m' is the least common multiple of numbers $m(x')$ and $m(x")$.

Theorem 4. *There exists a skew product $F \in \widetilde{T}^1_{*,1}(I)$ with a quotient map of type $\succ 2^\infty$ such that some its neighborhood $B^1_\varepsilon(F)$ in the space $\widetilde{T}^1_{*,1}(I)$ does not contain Ω-stable skew products of maps of an interval.*

The concept of dense stability as a whole of a family of fibers maps I

Definition 6. *A family of fibers maps of a skew product* $F \in \widetilde{T}^1_{*,j}(I)$ *(*$j = 2, 3, 4$*) with a quotient map of type* $\succ 2^\infty$ *is densely stable as a whole in* C^1*-norm* if there is an open everywhere dense in $\Omega(f)$ set $A(f) \subset \Omega(f)$, $A(f) \neq \Omega(f)$, such that for any $\delta > 0$ there is a neighborhood $B^1_\varepsilon(F)$ of F in $\widetilde{T}^1_*(I)$ such that for any map $\Phi \in B^1_\varepsilon(F) \bigcap \widetilde{T}^1_{*,j}(I)$ and any I^*_i ($i \geqslant i^*$ for some $i^* \geqslant i_*$) one can find δ-close to the identity map in C^0-norm homeomorphism

$$H^{<I^*_i>} : \overline{\eta}^F_{I^*_i}{}_{|A(f)} \rightarrow \overline{\eta}^\Phi_{I^*_i}{}_{|A(\varphi)} \qquad \left(H^{<I^*_i>}(x,\, y) = (h_1(x),\, h^{<I^*_i>}_{2,\,x}(y))\right)$$

satisfying the equalities

The concept of dense stability as a whole of a family of fibers maps II

$$h_{2,\,x}^{<I_i^*>}{}_{|\overline{\eta}^F_{I_i^*}{}_{|A(f)}(x)} \circ g_{x,\,I_i^*}{}_{|\overline{\eta}^F_{I_i^*}{}_{|A(f)}(x)}(y) =$$

$$\psi_{h_1(x),\,I_i^*}{}_{|\overline{\eta}^\Phi_{I_i^*}{}_{|A(\varphi)}(h_1(x))} \circ h_{2,\,x}^{<I_i^*>}{}_{|\overline{\eta}^F_{I_i^*}{}_{|A(f)}(x)}(y),$$

where $(x;\, y)$ is a point of the graph of a function $\overline{\eta}^F_{I_i^*}{}_{|A(f)}$ in I.

Existence theorem

Theorem 5. *There exists a map* $F_j \in \widetilde{T}^1_{*,j}(I)$ *for every* $j = 2, 3, 4$ *possessing densely stable as a whole in* C^1*-norm family of fibers maps.*

Theorem 6. *Any map* $F \in \widetilde{T}^1_{*,j}(I)$ $(j = 3\,or\,4)$ *with a quotient of type* $\succ 2^\infty$ *and densely stable as a whole in* C^1*-norm family of fibers maps such that for every* $x', x'' \in A(f)$ *and every* $i \geqslant i^*$ *fibers maps* $g_{x',I_i^*}, g_{x'',I_i^*}$ *belong to a same connected component* C_i *of the space* $C^1_\omega(I_2)$*, can be approximated up to an arbitrary accuracy with the use of* C^1*-smooth* Ω*-stable skew products of maps of an interval.*