Equations over groups and approximation of groups.

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- Free product $G_1 * G_2$.

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- There are some conjectures which are open. Kervaire conjecture....

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  - Locally indicable groups (James Howie).

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- Approximation is a sequence of almost-homomorphisms. We may approximate H by sequence of G (Gordon) or G by sequence of H.
- Here we will consider approximations of G, where G is a discrete group.

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- $\Phi \subset_{fin} G$, $\delta, \epsilon > 0$. We define (Φ, ϵ, δ)-homomorphism $\phi : \Phi \to H$ as follows
  - $\phi(1) = 1$ (if $1 \in \Phi$)
  - $d(1, \phi(y)) \geq \delta$ for all $y \in \Phi \setminus \{1\}$
  - $d(\phi(gh), \phi(g)\phi(h)) < \epsilon$, if $g, h, gh \in \Phi$

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- Sofic groups are hyperlinear.
- Weakly sofic group, $\mathcal{K}$ finite groups with all possible metrics.

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- The open question if there exists non-sofic (non-hyperlinear groups.)

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- It does not look so useful but...

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- If $\mathcal{K}$ is a class of some AGACF then $\mathcal{K}$-approximability implies *Fin*-approximability.