On interrelation between dynamics and topology of ambient manifold

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Goal of the lectures

The main goal is to present some results on interrelations between topology of $M^3$ and dynamics of Morse-Smale systems acting on it. We will give sketches of proofs of the statements below and discuss their applications.
Morse-Smale systems

Morse-Smale vector fields were introduced by S. Smale (in the paper “Morse inequality for Dynamical Systems” Bull. Amer. Math. Soc. 1960, No. 66, 46-49) as smooth vector fields $X$ on smooth $n$-manifold $M^n$ with following properties:

1. $X$ has a finite number of hyperbolic singular points $\beta_1, \ldots, \beta_k$;

2. $X$ has a finite number of hyperbolic closed orbits $\beta_{k+1}, \ldots, \beta_m$;

3. the stable and unstable manifolds of the $\beta_i$ have transversal intersection with each other.
Morse-Smale diffeomorphisms

Let $M^n$ is an orientable smooth closed manifold of dimension $n$.

**Definition**

Diffeomorphism $f : M^n \rightarrow M^n$ is a *Morse-Smale diffeomorphisms* if:

- it’s nonwandering set $\Omega(f)$ consists of finite number of hyperbolic periodic points,
- stable and unstable manifolds of different points from $\Omega(f)$ have transversal intersection.
Heteroclinic intersections for n=3

Now suppose that \( n = 3 \) and \( p, q \in \Omega(f) \) be saddle points. A point \( x \in W^u(p) \cap W^s(q) \) is called heteroclinic point if \( \dim W^u(p) + \dim W^s(q) = 3 \).

If intersection \( W^u(p) \cap W^s(q) \) is not empty and \( \dim W^u(p) = \dim W^s(q) = 2 \) then each connected component of \( W^u(p) \cap W^s(q) \) is called heteroclinic curve.
Gradient-like systems

Morse-Smale vector field is called gradient-like if it does not contain closed orbit. Morse-Smale diffeomorphisms is called gradient-like if condition

\[ W^u(p) \cap W^s(q) \neq \emptyset \] implies \( \dim W^s(p) < \dim W^s(q) \)

Remark

When \( n = 3 \), a Morse-Smale diffeomorphism is gradient-like if and only if the two-dimensional and one-dimensional invariant manifolds of it’s different saddle points do not intersect. Let us emphasize that two-dimensional invariant manifolds of different saddle points of a gradient-like system may have a non empty intersection along heteroclinic curves.
A manifold $M^3$ is called connected sum of manifolds $M^3_1$ and $M^3_2$:

$$M^3 = M^3_1 \# M^3_2$$

if $M^3$ is obtained by choosing 3-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and by gluing the manifolds $M_1 \setminus B_1$ and $M_2 \setminus B_2$ by a diffeomorphism between the boundaries which reverses the natural orientation on the boundaries.
Decomposition Theorem

A manifold $M^3$ is called **irreducible** if any 2-dimensional sphere regularly embedded in $M^3$ bounds 3-dimensional disk. The manifold $M^3$ is said **simple** if $M^3$ is closed and if $M^3$ is either irreducible or homeomorphic to $S^2 \times S^1$.

Kneser, Milnor

Any orientable closed manifold $M^3$ can be decomposed into the connected sum of simple manifolds and the decomposition is unique.
Let $M^3$ be a three-dimensional closed, connected, orientable manifold. Set
\[ g = \frac{k - \ell + 2}{2}, \]
where $k$ is the number of saddles and $\ell$ is the number of sinks and sources of gradient-like flow (Morse-Smale diffeomorphisms) given on $M^3$.

**Theorem (Ch. Bonatti, V. Grines, V. Medvedev, E. Pécou.)**

There exists a gradient-like flow (Morse-Smale diffeomorphism) without heteroclinic trajectories (curves) on $M^3$ if and only if $M^3$ is the sphere $S^3$ and $k = \ell - 2$, or $M^3$ is the connected sum of $g$ copies of $S^2 \times S^1$. 
The theorem above is sufficient condition of existence of a heteroclinic curve for given Morse-Smale diffeomorphism.

- For example, if $\Omega(f)$ of diffeomorphism $f : S^3 \to S^3$ consists of two saddles, one sink and one source then wandering set of such diffeomorphism contains a heteroclinic curve. Moreover, in this case there is at least one noncompact heteroclinic curve.

- If ambient manifold of Morse-Smale diffeomorphism $f$ is not homeomorphic to the connected some of copies of $S^2 \times S^1$ (for example if ambient manifold is the torus $T^3$) then $\Omega(f)$ contains at least one heteroclinic curve.

Lemma

Let wandering set of gradient-like flow (Morse-Smale diffeomorphism) on $M^3$ does not contain heteroclinic curve. Then there is a saddle periodic point $p \in \Omega(f)$ such that either $W^u(p)$ or $W^s(p)$ has dimension 2 and does not contain heteroclinic points.

If we suppose the contrary we come to the contradiction with finiteness of the nonwandering set $\Omega(f)$. 
On existence of invariant sphere which is smooth everywhere except one source or sink

Suppose for definiteness that there is point \( p \in \Omega(f) \) such that \( W^s(p) \) has dimension 2 and does not contain heteroclinic points. As by assumption \( W^s(p) \) does not contain heteroclinic curves then by S. Smale there is a source \( \alpha \in \Omega(f) \) such that \( \Sigma = W^s(p) \cup \{ \alpha \} \) is homeomorphic to the sphere \( S^2 \).
On existence of $f^{-m}$-compressed smooth annulus

**Lemma**

There is an open domain $K$ such that:

- $\Sigma \subset K$;
- $\overline{K} = K \cup S_1^2 \cup S_2^2$;
- $\overline{K}$ is diffeomorphic to $S^2 \times [0, 1]$;
- There is $m > 0$ such that $f^{-m}(\overline{K}) \subset K$. 
Pixton example with wild separatrices (1977) Knot invariants by Bonatti-Grines (2000)

$f, f' : \mathbb{S}^3 \to \mathbb{S}^3$ have isomorphic graphs and don’t topologically conjugated, because separatrices of $\sigma'$ are wild. There are series results obtained by C. Bonatti, V. Grines, V. Medvedev, E. Pécou and O. Pochinka in (2000-2010) on classification of Morse-Smale diffeomorphisms under suggestions of different generality. O. Pochinka got complete classification of arbitrary 3D-Morse-Smale diffeomorphisms in 2011.
On existence a “good” neighborhood for a sphere smooth everywhere except one point

Lemma

Let $\eta : S^2 \to M^3$ be a topological embedding which is a smooth everywhere except at one point and let $\Sigma = \eta(S^2)$. Then any neighborhood of $\Sigma$ contains a subneighborhood $K$ which is diffeomorphic to $S^2 \times [0, 1]$. 
Remark

If $\eta : S^2 \to M^3$ be a topological embedding which is a smooth everywhere except at least two points then lemma above is not true in general and consequently it is non trivial topological fact.
Cut and glue operations. Finding of a copy $S^2 \times S^1$

The boundary of $M^3 \setminus K$ consists of two connected components diffeomorphic to the sphere $S^2$. Let us glue to these connected components the boundaries of smooth 3-balls and denote by $\tilde{M}^3$ new manifold. There are two possibilities:

1) $\tilde{M}^3$ is connected and in this case $M^3$ is homeomorphic to $\tilde{M}^3 \# (S^2 \times S^1)$;

2) $\tilde{M}^3$ consists of two connected components $\tilde{M}^3_1$, $\tilde{M}^3_2$ and in this case $M^3$ is homeomorphic to $\tilde{M}^3_1 \# \tilde{M}^3_2$. 

![Diagram](image_url)
Construction of new Morse-Smale diffeomorphism without one saddle point

Without lost of generality we can suppose that \( \Omega(f) \) consists of fixed points.

In both cases above one can construct a Morse-Smale diffeomorphism \( \tilde{f} : \tilde{M}^3 \to \tilde{M}^3 \) whose nonwandering set contains \( k - 1 \) saddle fixed points and \( \ell + 1 \) sinks and sources.

Let us continue the process of cutting of the manifold \( \tilde{M}^3 \) and then cutting of the manifold that we will get and so on.
Two kinds of saddles

- Let $g$ be number of saddles for which closer of two-dimensional unstable or stable manifold does not cut appropriate manifold on two connected components.

In this case after cutting of anulus neighborhood and gluing balls we find a copy of $S^2 \times S^1$ in decomposition of manifold $M^3$.

- Then $k - g$ is the number of saddles whose closer of two-dimensional unstable or stable manifold cuts appropriate manifold on two connected components.
Calculation of the number of saddles which define all copies $S^2 \times S^1$ in decomposition $M^3$

After all cuts we get the manifold which admits Morse-Smale diffeomorphism whose nonwandering set consists of only sinks and sources. Consequently this manifold is the union of $k - g + 1$ copies of 3-spheres. As each sphere contains exactly one sink and one source we get $k - g + 1 = \frac{k+\ell}{2}$.

It follows from this equation that

$$g = \frac{k - \ell + 2}{2}$$

and consequently $M^3$ is the connected sum of $g = \frac{k-\ell+2}{2}$ copies of $S^2 \times S^1$. That is the theorem is proved.
Lemma

Let $g$ be non negative integer, and let $M^3$ be the 3-sphere if $g = 0$, or the connected sum of $g$ copies of $S^2 \times S^1$ if $g > 0$. Then for any non-negative integers $r$ and $l$ such that $r = l + 2g - 2$, there exists a Morse-Smale vectorfield without heteroclinic curve and without periodic trajectories on $M^3$ admitting $r$ saddle points and $l$ sinks and sources.
Sketch of a proof. The case of the sphere

First consider the case of the sphere: take any non-negative integer $r$ and set $r = k + 2$. There is a Morse-Smale flow $X_0$ on the compact 3-ball $B$ pointing inward transversally to the sphere $S = \partial B$ and having exactly $k + 1$ sinks and $k$ saddles having a 2-dimensional stable manifold (and no periodic orbit): the attracting set in the ball $B$ is a smooth compact arc formed by the sinks and by the unstable manifolds of the saddle points.
The case of the sphere

Denote by $X_1$ a flow on the ball $B$ pointing outward transversally to $S$ and having a unique source inside $B$ (and no periodic orbit). By gluing two copies of the ball $B$ one endowed with $X_0$ and the other endowed with $X_1$ we get the 3-sphere $S^3$ with a Morse-Smale vectorfield without heteroclinic curves and without periodic trajectories and having exactly $l$ sources and sinks and $r$ saddles.
Handlebodies

A three-dimensional orientable manifold is called a handlebody of a genus $g \geq 0$ if it is obtained from a 3-ball by an orientation reversing identification of $g$ pairs of pairwise disjoint 2-discs in its boundary. The boundary of such a handlebody is an orientable surface of genus $g$. 
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The case when $M^3$ is the connected sum of $g > 0$ copies of $\mathbb{S}^2 \times \mathbb{S}^1$

Notice that $M^3$ is obtained by gluing two copies of the handlebody $B_g$ of genus $g$ by a diffeomorphism of its boundary $S_g = \partial B_g$ which is isotopic to identity.
Choose disjoint discs $d_1, \ldots, d_g$ which are transversal to the boundary $B_g$ and such that $B_g$ minus union of all disks will be homeomorphic to 3-ball. Denote by $\gamma_i$ the boundary of $d_i$. As $g = \frac{r - \ell + 2}{2}$ and $\ell \geq 2$ then $r \geq 2g$. Represent $r = r_1 + r_2$ where $r_j \geq g, j = 1, 2$. Construct vectorfields $X_{r_1}, X_{r_2}$ such that:
1) $X_{r_1}$ ($X_{r_2}$) is pointing inward (outward) transversally to $S_g$;
2) $X_{r_1}$ ($X_{r_2}$) has exactly $r$ saddles and $r_1 - g + 1$ sinks ($r_2 - g + 1$ sources) and has no periodic trajectories;
3) for any saddle $\sigma$ of $X_{r_1}$ ($X_{r_2}$) stable manifold $W^s_\sigma$ (unstable manifold $W^u_\sigma$) is compact disk transversal to $S_g$ and $W^s_\sigma \cup S_g$ ($W^u_\sigma \cup S_g$) is closed curve which is homotopic to one curves $\gamma_i$;
4) for any $i$ there is saddle $\sigma$ of $X_{r_1}$ ($X_{r_2}$ such that $W^s_{\sigma_i} \cup S_g$ ($W^u_{\sigma_i} \cup S_g$) is homotopic to $\gamma_i$
Finally we glue the vectorfield $X_{r_1}$ with the vectorfield $X_{r_2}$ using a diffeomorphism $\phi$ and we get a Morse-Smale vectorfield $X$ without heteroclinic curves and without periodic orbits on a closed 3-manifold diffeomorphic to $M^3$. 
Moreover $X$ has exactly $r_1$ saddles with 2-dimensional stable manifolds, $r_2$ saddles with 1-dimensional stable manifolds, $l_1 = k_1 - g + 1$ sinks and $l_2 = k_2 - g + 1$ sources.
Topology of ambient manifold $M^3$ admitting gradient-like flows and diffeomorphisms

Let

$$g = \frac{k - \ell + 2}{2},$$

where $k$ is the number of saddles and $\ell$ is the number of sinks and sources of gradient-like flow (diffeomorphisms) given on $M^3$.

**Theorem (V. Grines, V. Medvedev, E. Zhuzhoma)**

If $M^3$ admits gradient-like flow (diffeomorphism with tame bunches of one-dimensional separatrices of saddles) then the manifold $M^3$ admits the Heegaard splitting of genus $g$. 
Handlebodies

A three-dimensional orientable manifold is called a handlebody of a genus $g \geq 0$ if it is obtained from a 3-ball by an orientation reversing identification of $g$ pairs of pairwise disjoint 2-discs in its boundary.
The boundary of such a handlebody is an orientable surface of genus $g$. 
Handlebodies

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A Heegaard splitting of genus $g \geq 0$ for a manifold $M^3$ is a representation of $M^3$ as the gluing of two handlebodies of genus $g$ by means of some diffeomorphism of their boundaries. Their common boundary after gluing, a surface of genus $g$ in $M^3$, is called a Heegaard surface.
Heegaard splitting

Let $V$ and $W$ be handlebodies of genus $g$, and let $f$ be an orientation reversing homeomorphism from the boundary of $V$ to the boundary of $W$. By gluing $V$ to $W$ along $f$ we obtain the closed oriented 3-manifold $M^3 = V \cup_f W$.

The decomposition of $M^3$ into two handlebodies is called a Heegaard splitting, and their common boundary $S_g$ is called the Heegaard surface of the splitting.

A Heegaard splitting is minimal or minimal genus if there is no other splitting of the ambient three-manifold of lower genus. The minimal value $g$ of the splitting surface is the Heegaard genus of $M^3$. We denote this number $G(M)$ ($g \geq G(M)$).
Topology of ambient manifold $M^3$ admitting gradient-like flows and diffeomorphisms

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Application for finding of closed trajectories of Morse-Smale flows on $M^3$

Let $f^t$ be Morse-Smale flow on $M^3$ with

$$g = \frac{k - \ell + 2}{2}$$

and $G(M) > g$ then $f^t$ has at least one closed trajectori.
Bunches of separatrices

Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism and $L(\omega)$ be the bunch of all unstable one-dimensional separatrices of saddles which contain $\omega$ in their closure.

A bunch $L(\omega)$ is called **tame** if there is a homeomorphism $\psi : W^s(\omega) \to \mathbb{R}^3$ such that $\psi(\omega) = O$, where $O$ is the origin and $\varphi(\ell \setminus \sigma)$ is a ray starting from $O$ for any separatrix $\ell \in L(\omega)$. In the opposite case the set $L(\omega)$ is **wild**.

Notice that the tameness of each separatrix $\ell \in L(\omega)$ does not imply the tame property of the bunch $L(\omega)$. Such bunch is called **mildly wild**.
So H. Debrunner and R. Fox in 1960 constructed an example of a wild collection of arcs in $\mathbb{R}^3$ where each arc is tame.
Mildly wild bunch in dynamics

Using this example and methods of realization of Morse-Smale diffeomorphisms it is possible to construct a gradient-like diffeomorphism on $S^3$ having mild wild bunch $L(\omega)$ (O. Pochinka).
Global dynamic of gradient-like diffeomorphism.
One-dimentional attractor and repeller

Let $f$ be a Morse-Smale diffeomorphism on 3-manifold. Let us denote by $\Omega^+$ ($\Omega^-$) the set of all sinks (sources), by $\Sigma^+$ ($\Sigma^-$) the set of all saddle points having one-dimensional unstable (stable) invariant manifolds, by $L^+$ ($L^-$) the union of the unstable (stable) one-dimensional separatrices.

We set $\mathcal{A} = \Omega^+ \cup L^+ \cup \Sigma^+$, $\mathcal{R} = \Omega^- \cup L^- \cup \Sigma^-$. It is possible to prove that $\mathcal{A}$ ($\mathcal{R}$) is a connected set which is an attractor (a repeller) of $f$. Then $g = \frac{|\Sigma^+ \cup \Sigma^-| - |\Omega^+ \cup \Omega^-| + 2}{2}$, where $|.|$ stands for the cardinality.
Number characteristic for one-dimensional attractor and repeller

Set $g^+ = |\Sigma^+| - |\Omega^+| + 1$ and $g^- = |\Sigma^-| - |\Omega^-| + 1$.

**Statement**

*For any Morse-Smale diffeomorphism $f : M^3 \to M^3$*

$$g^+ = g^- = g.$$

**Step 1.** According to M. Shub and D. Sullivan, a Morse-Smale diffeomorphism induces in all homology groups isomorphisms whose eigenvalues are roots of unity. Thus there is an integer $k$ such that $f^k$ leaves $\text{Per}(f^k)$ fixed, $f^k|_{W^u(p)}$ preserves the orientation of $W^u(p)$ for any point $p \in \text{Per}(f^k)$ and 1 is the only eigenvalue of the isomorphism induced by $f^k$ on homology.
Number characteristic for one-dimensional attractor and repeller

Step 2. Applying the Lefschetz formula to $f^k$ yields

$$\sum_{p \in \text{Per}(f^k)} (-1)^{\dim W^u(p)} = \sum_{i=0}^{3} (-1)^i t_i,$$

where $t_i$ is the trace of the map induced by $f^k$ on the $i$-th homology group $H_i(M, \mathbb{R})$. By assumption on $k$, $t_i$ coincides with the $i$-th Betti number and the alternating sum of the $t_i$'s is the Euler characteristic, which is 0 for $M^3$. So we get

$$|\Omega^+| - |\Sigma^+| + |\Sigma^-| - |\Omega^-| = 0,$$

hence

$$|\Sigma^-| - |\Omega^-| = |\Sigma^+| - |\Omega^+|,$$

that is $g^+ = g^-$. As $g^+ = |\Sigma^+| - |\Omega^+| + 1$,

$g^- = |\Sigma^-| - |\Omega^-| + 1$ then

$g^+ + g^- = |\Sigma^+| - |\Omega^+| + 1 + |\Sigma^-| - |\Omega^-| + 1 = k - l + 2 = 2g$. Then we get $g^+ = g^- = g$. 

Neighborhoods of attractor and repeller

**Theorem**

There is a neighborhood $P^+$ ($P^-$) of the set $\mathcal{A}$ ($\mathcal{R}$) such that:

1. $P^+$ (resp. $P^-$) is a $f$-compressed (resp. $f^{-1}$-compressed) handlebody of genus $g$ and $\mathcal{A} \subset P^+$ (resp. $\mathcal{R} \subset P^-$);

2. $W^s(\sigma^+) \cap P^+$ (resp. $W^u(\sigma^-) \cap P^-$) consists of exactly one two-dimensional closed disk for each saddle point $\sigma^+ \in \Sigma^+$ (resp. $\sigma^- \in \Sigma^-$).
Construction of $P^+$ ($P^-$)

- It follows from suggestion that all bunches of one-dimensional separatrices $L^+$ are tame that there exists an $f$-compressed domain $B^+$, made of $|\Omega^+|$ balls, which is a neighborhood of $\Omega^+$ and such that each separatrix $\ell \in L^+$ intersects $\partial B^+$ at unique point.

- Choose a tubular neighborhood $H^+$ of $L^+ \setminus \text{int } B^+$ such that $P^+ = B^+ \cup H^+$ is $f$-compressed. Then $P^+$ is a handlebody of genus $g$. Indeed, $P^+$ has homotopic type of cellular complex with $|\Omega^+|$ 0-cells and $|\Sigma^+|$ 1-cells. Then $\beta_0 - \beta_1 = |\Omega^+| - |\Sigma^+|$, where $\beta_0$, $\beta_1$ are the Betti numbers of $P^+$. Take account of that $\beta_0 = 1$ and $\beta_1$ is the genus of $P^+$, we get that genus of $P^+$ equals $|\Sigma^+| - |\Omega^+| + 1 = g^+ = g.$
Lemma

Let $Q$ be an orientable surface with following properties:

- $Q$ is smoothly embedded to interior of manifold $H = S_g \times [0, 1]$ where $S_g$ is orientable surface of genus $g$;
- $Q$ does not bound an open domain.

Then the closer of each component of the set $H \setminus Q$ is homeomorphic to $H$. 
Construction of Heegaard splitting of genus $g$

- Denote by $S^+ (S^-)$ the boundary of handlebody $P^+ (P^-)$. It follows from above that $S^+$ and $S^-$ are smooth orientable closed surfaces of genus $g$.

- As the sets $L(\omega), L(\alpha)$ are tame for any $\omega, \alpha \in \Omega(f)$ then both sets $P^+ \setminus \mathcal{A}, P^- \setminus \mathcal{R}$ are homeomorphic to the product $S_g \times (0, 1]$ where $S_g$ is orientable surface of genus $g$. 
Construction of Heegaard splitting of genus \( g \)

- Choose a number \( N > 0 \) such that \( f^N(S^-) \subset \text{int } P^+ \) and put \( S_N^- = f^N(S^-) \).
Construction of Heegaard splitting of genus $g$

- Choose a surface $S^+_N \subset P^+ \setminus A$ such that:
  1. $S^+_N$ is leaf which appropriates to some value $t_N \in (0, 1)$ in the product $S_g \times (0, 1]$.
  2. $S^-_N$ belongs to the interior of closed domain which is homeomorphic to the product $S_g \times [0, 1]$ and bounded by $S^+_N$ and $S^+$.
Construction of Heegaard splitting of genus \( g \)

- By splitting lemma the closed set bounded by \( S^+_N \) and \( S^-_N \) also is homeomorphic to the product \( S^+_g \times [0, 1] \).
- Then the closed set \( M^+_N \) (or \( M^-_N \)) which contains the attractor \( A \) (the repeller \( R \)) and bounded by surface \( S^-_N \) is a handlebody of genus \( g \).
- Consequently \( M^3 = M^+_N \cup M^-_N \) is a Heegaard splitting of genus \( g \).
The results was obtained in collaboration with Ch. Bonatti, V. Grines, V. Medvedev, E. Pécou, O. Pochinka, E. Zhuzoma