On chaotic dynamics in the nonholonomic Suslov model

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Problem statement

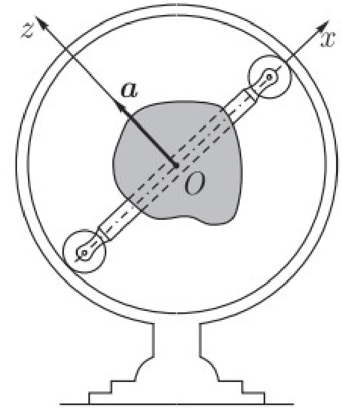
- A motion of a body with fixed point subjected to nonholonomic constraint

$(\boldsymbol{\omega}, \mathbf{a}) = 0$, where

$\boldsymbol{\omega}$ - the vector of angular velocity

$\mathbf{a}$ - the uniform vector fixed in a body

$$\mathrm{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}$$ - inertia tensor



G. Wagner's implementation (1941)

Problem statement

We introduce the coordinate system Oxyz with origin in the fixed point of a body such a way that:

- $Oz \parallel \mathbf{a}$
- Ox, Oy align such a way that $I_{12} = 0$

In Oxyz the nonholonomic constraint takes the form

$$\omega_3 = 0$$

The body moves in the potential field $U = (\mathbf{b}, \gamma),$

where **b –** the constant vector in the Oxyz

Equation of motion

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$$I_{11}\dot{\omega}_1 = -\omega_2(I_{13}\omega_1 + I_{23}\omega_2) + b_3\gamma_2 - b_2\gamma_3$$

$$I_{22}\dot{\omega}_2 = \omega_1(I_{13}\omega_1 + I_{23}\omega_2) + b_1\gamma_3 - b_3\gamma_1$$

$$\dot{\gamma}_1 = -\gamma_3\omega_2$$

$$\dot{\gamma}_2 = \gamma_3\omega_1$$

$$\dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1$$

$$I_{11}, I_{22}, I_{13}, I_{23}, b_1, b_2, b_3$$

First integrals

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$$E = \frac{1}{2}(I_{11}\omega_1^2 + I_{22}\omega_2^2) + b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3$$

$$F_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

First integrals

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$$I_{13} = I_{23} = 0$$

$$b_1 = b_2 = b_3 = 0$$

$$I_{13} = I_{23} = 0 \qquad b_3 = 0$$

Reversibility and involution

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$$\omega_1 \to -\omega_1, \quad \omega_2 \to -\omega_2, \quad t \to -t$$

$$I_{13} = 0$$

$$\omega_1 \to -\omega_1, \quad \gamma_1 \to -\gamma_1, \quad t \to -t$$

$$I_{23} = 0$$

$$\omega_2 \to -\omega_2, \quad \gamma_2 \to -\gamma_2, \quad t \to -t$$

$$I_{13} = I_{23} = 0$$

Poincare map

- On the common level set of 2 integrals(energy and geometrical) the dynamics of the system can be described by 3D flow.
- We take $\gamma_1 = 0$ as a section
- and construct 2D Poincare map in the variables

$$(\gamma_2, \omega_1)$$

Numerics

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$$I_{11} = 3, I_{22} = 4, I_{13} = 0,$$

$$b_1 = 0, b_2 = 0, b_3 = 100$$

$$I_{23}$$

$$\gamma_1 = 0$$

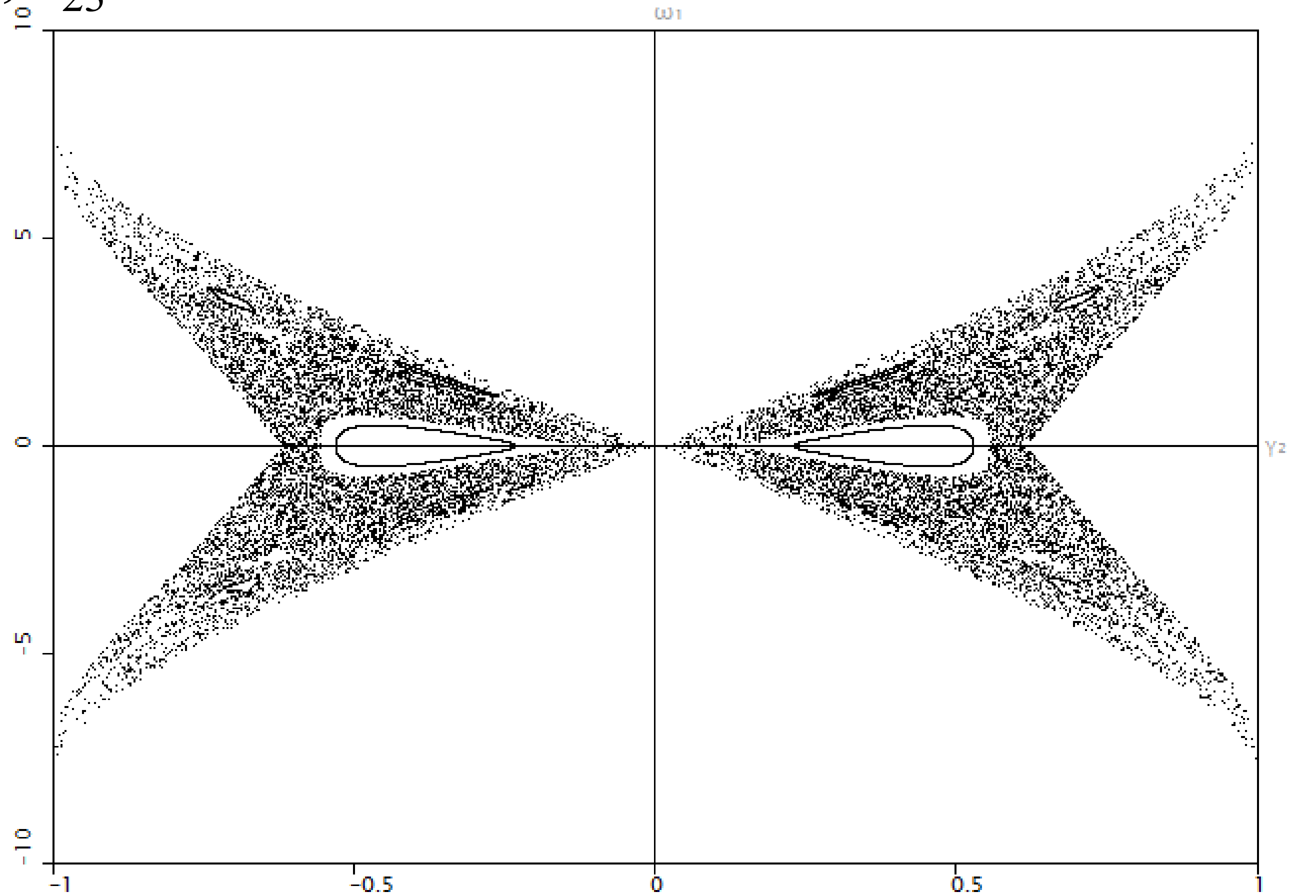
$$(\gamma_2, \omega_1)$$

$$\omega_1 \to -\omega_1$$

$$\omega_1 = 0$$

Numerics

$E = 100, I_{23} = 0$ (the system admits an invariant measure)



conservative chaos

Numerics. Chart of Lyapunov exponents

Lyapunov exponents: $l_1 \geq l_2 \geq l_3, L = l_1 + l_2 + l_3$

$l_1 < 0 \quad L < 0$

$l_1 \approx 0 \quad L < 0$

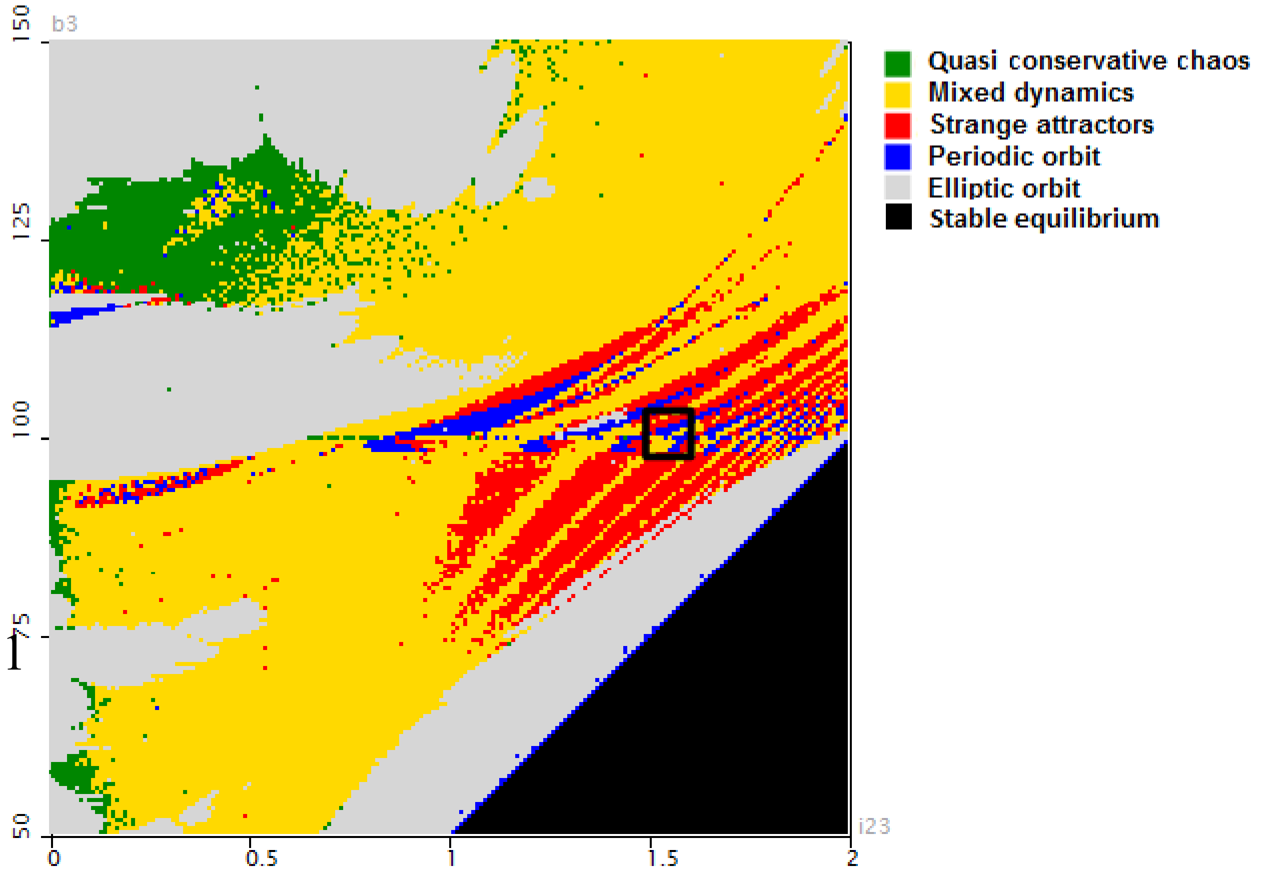
$l_1 \approx 0 \quad L = 0$

Chaos

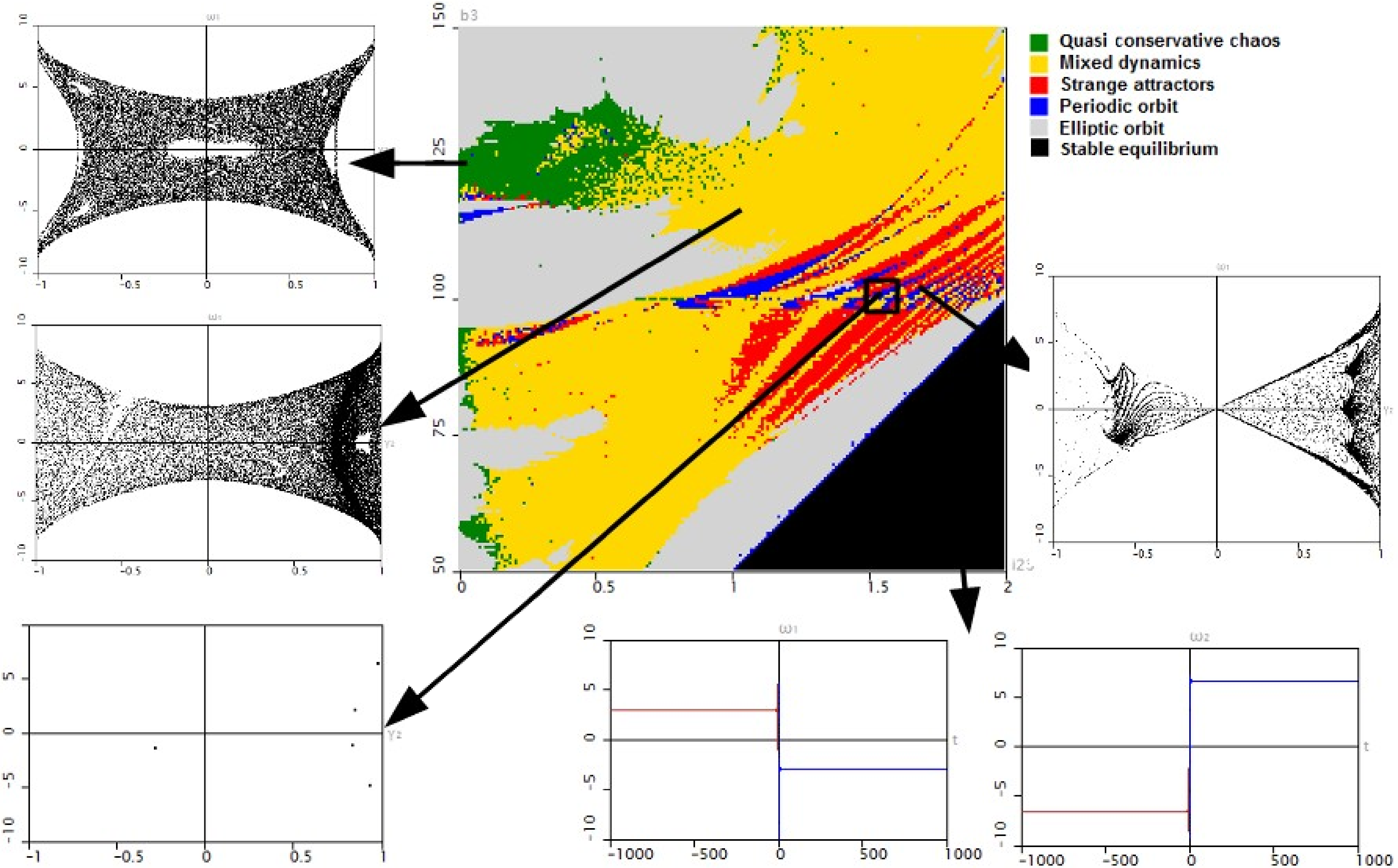
$l_1 > 0 \quad |L| < 0.0001$

$l_1 > 0 \quad 0.0001 < |L| < 0.01$

$l_1 > 0 \quad 0.01 < |L|$



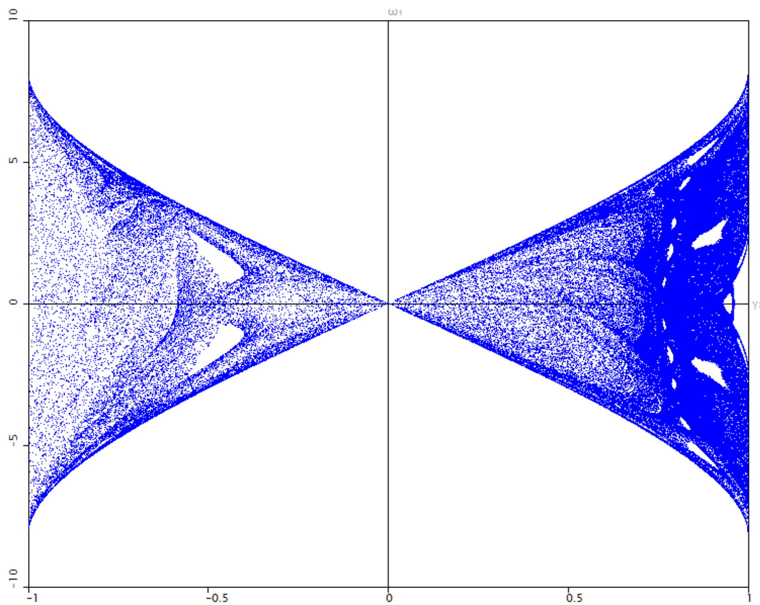
Numerics

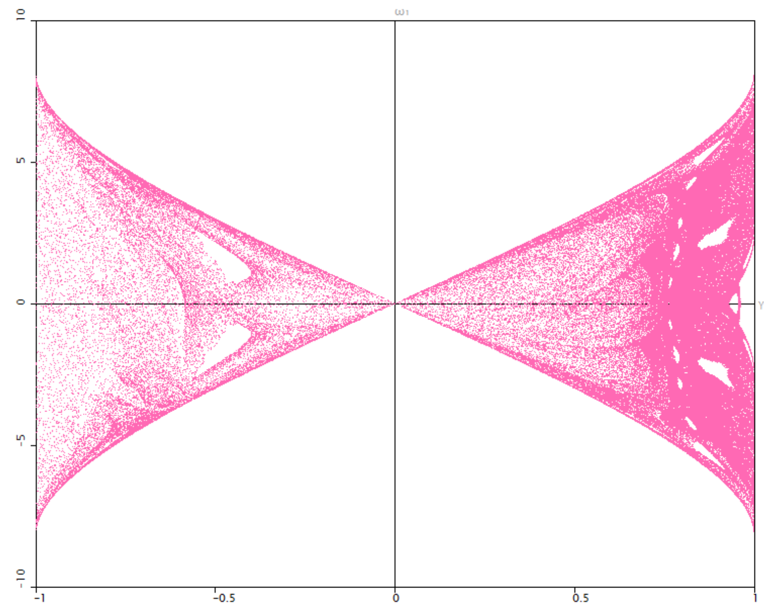

Quasi conservative chaos
Mixed dynamics
Strange attractors
Periodic orbit
Elliptic orbit
Stable equilibrium

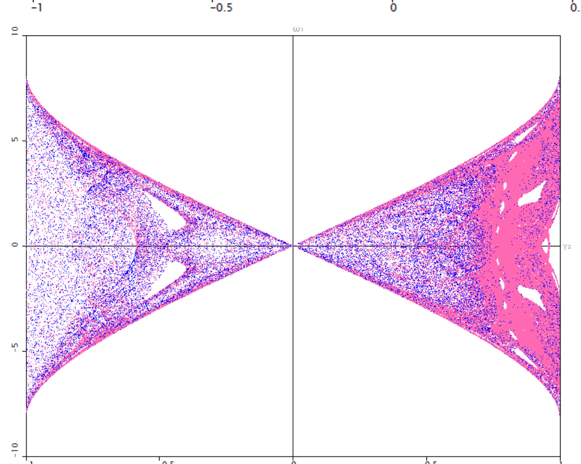
Numerics

$$I_{23} = 1.5, E = 100$$

Mixed dynamics. Attractor and repeller







Lyapunov exponents

$$l_1 = 0.0875$$

$$l_2 = -0.0005$$

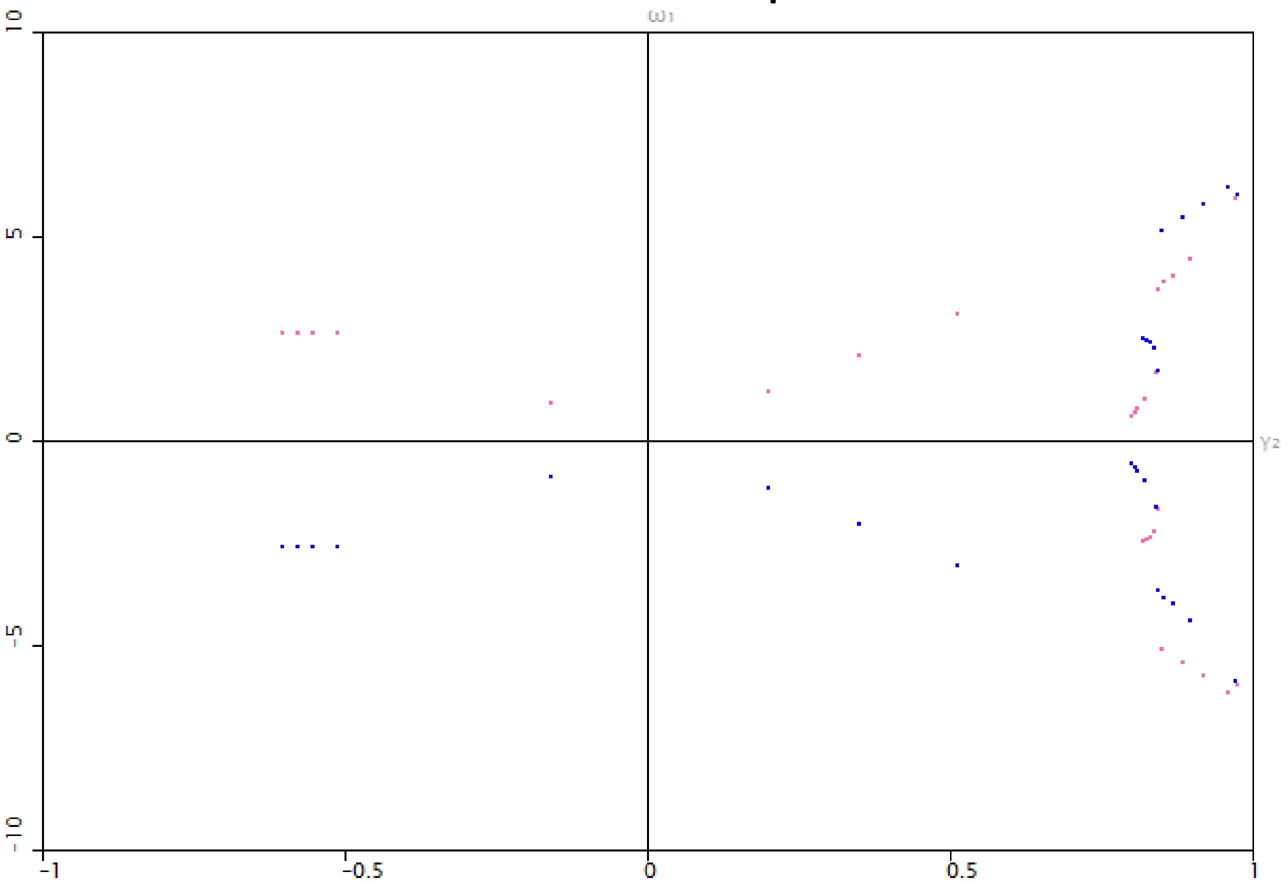
$$l_3 = -0.0912$$

$$l_1 + l_2 + l_3 = -0.0038$$

Numerics

$I_{23} = 1.5898, E = 100$

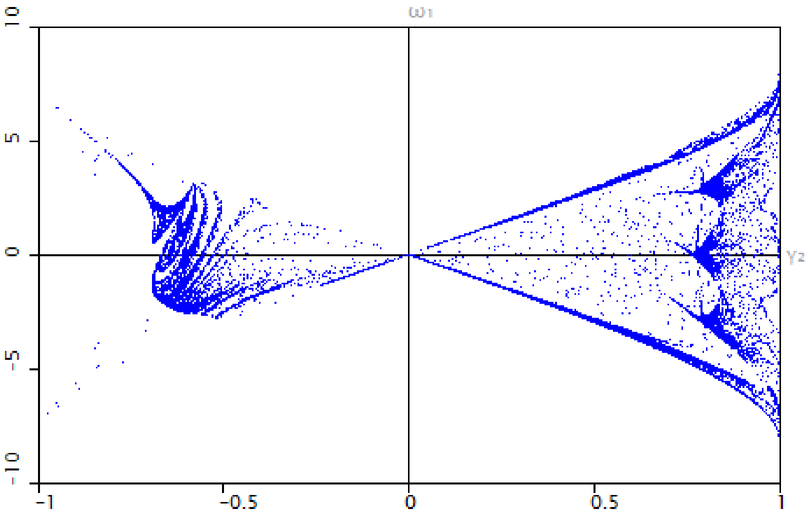
Stable and unstable orbit of period 18

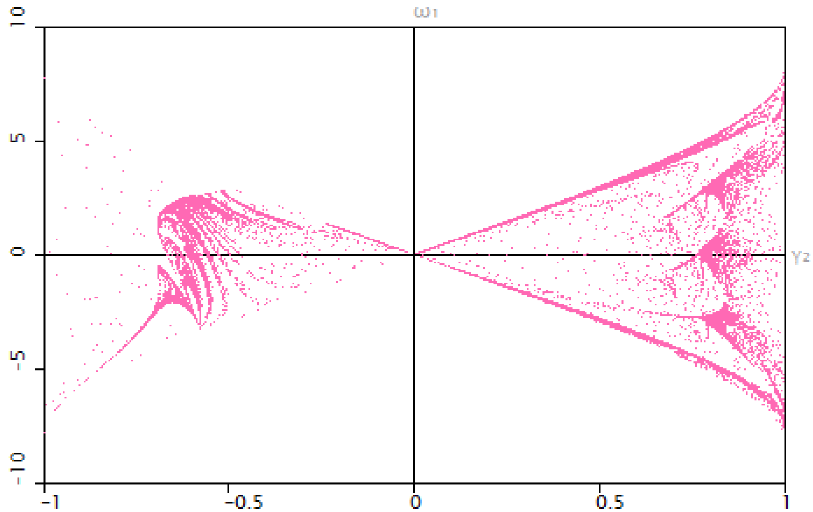


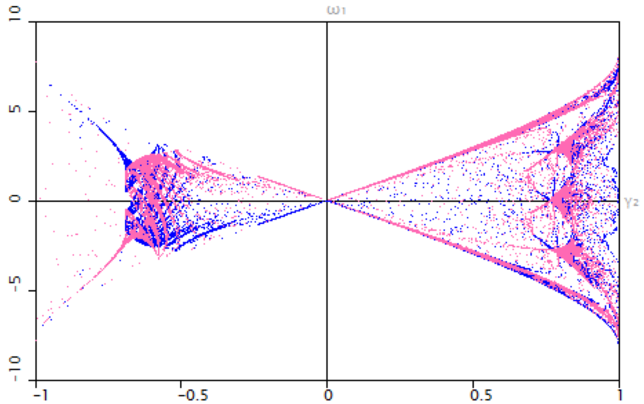
Numerics

$I_{23} = 1.5895, E = 100$

Strange attractor and strange repeller







Lyapunov exponents

$l_1 = 0.0663$

$l_2 = -0.008$

$l_3 = -0.178456$

$l_1 + l_2 + l_3 = -0.1146$

Equilibria

For some values of parameters I_{23}, E there is a stable equilibrium $\omega_1^*, \omega_2^*, \gamma_1^*, \gamma_2^*, \gamma_3^*$ in the system

Due to the involution $\omega_1 \to -\omega_1, \quad \omega_2 \to -\omega_2, \quad t \to -t$ there exists also an unstable equilibrium $-\omega_1^*, -\omega_2^*, \gamma_1^*, \gamma_2^*, \gamma_3^*$

And due to the additional involution

$$\omega_1 \to -\omega_1, \quad \gamma_1 \to -\gamma_1, \quad t \to -t$$

there exist a pair of stable and unstable equilibria:

$$-\omega_1^*, \omega_2^*, -\gamma_1^*, \gamma_2^*, \gamma_3^*$$

$$\omega_1^*, -\omega_2^*, -\gamma_1^*, \gamma_2^*, \gamma_3^*$$

Reverse

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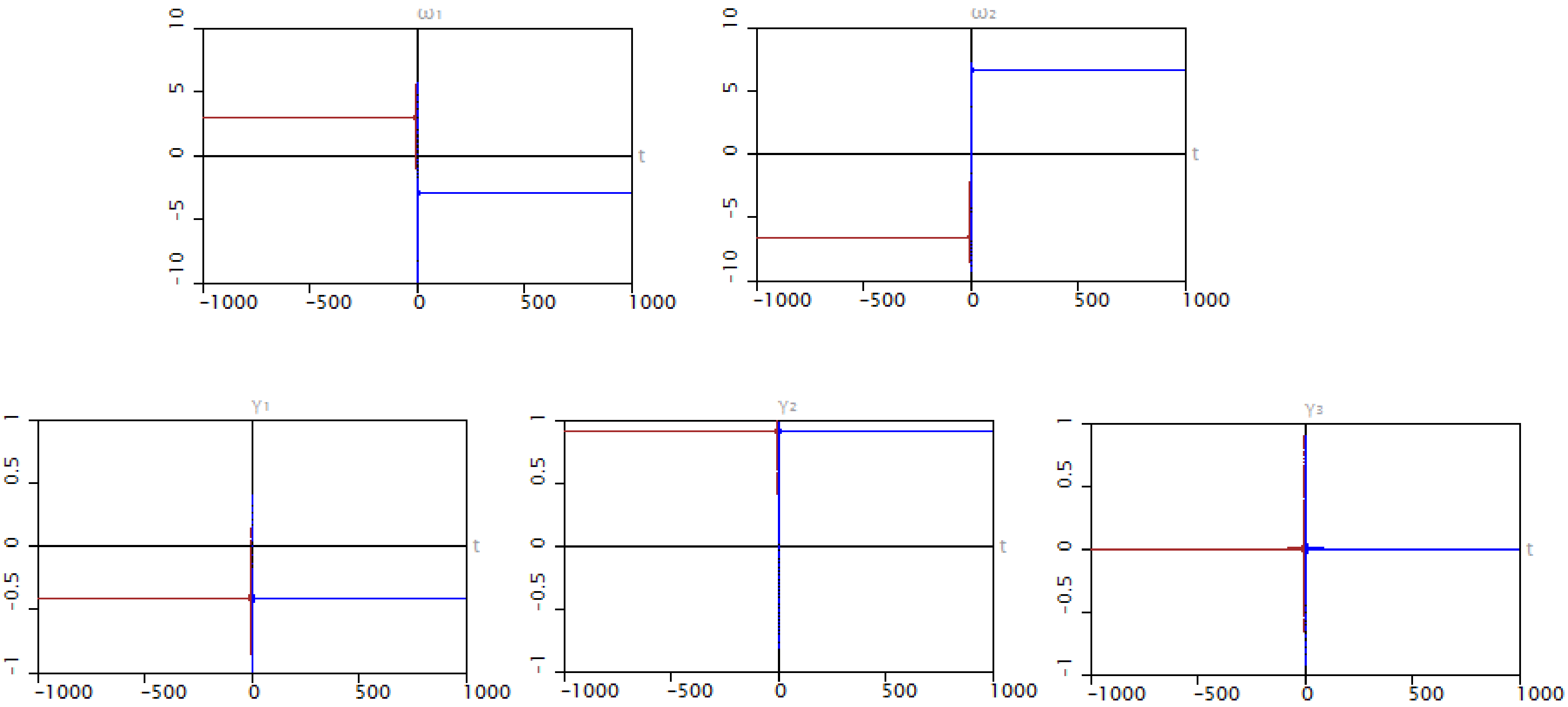
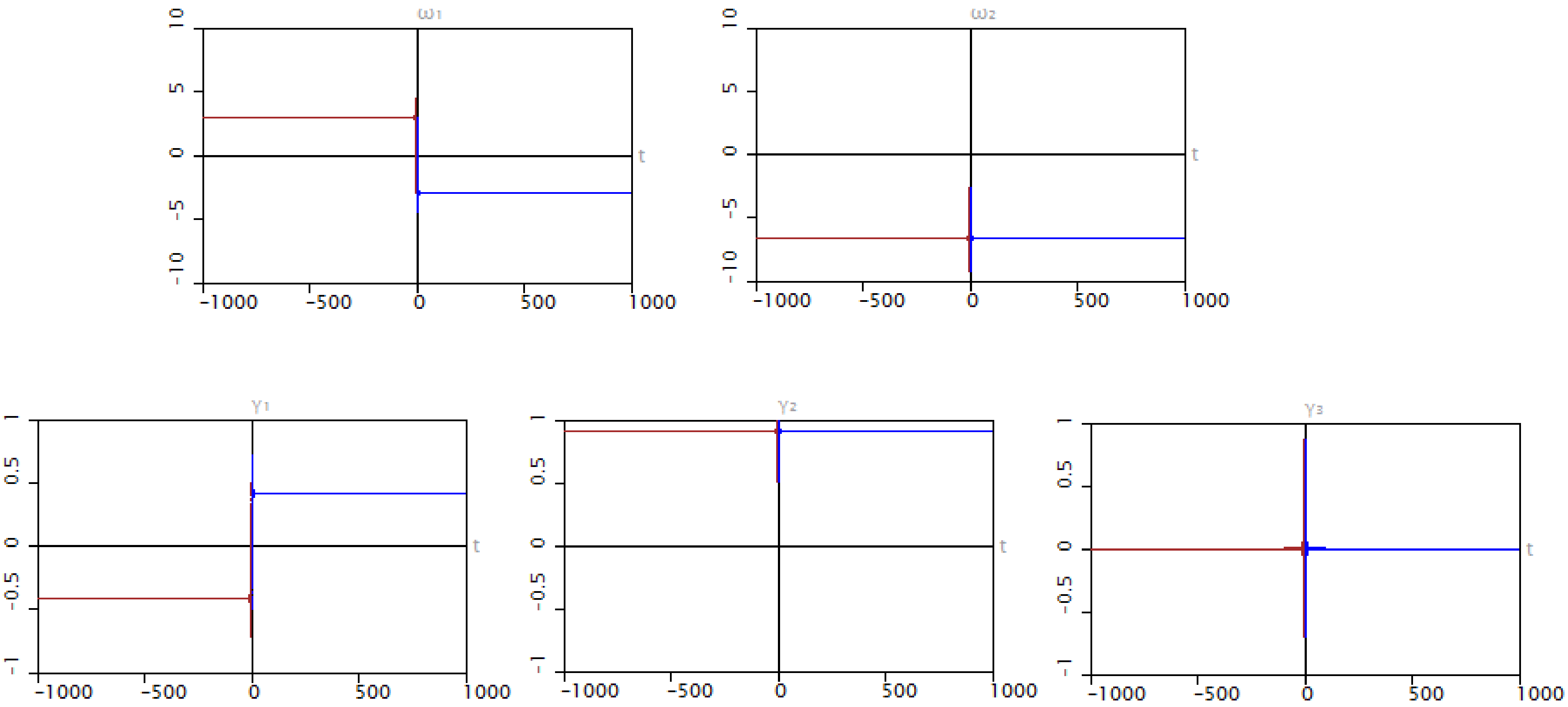
Reverse 1



Reverse 2



Thank you for your attention!