Hyperbolic chaos in model systems with ring geometry

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To Smale-Williams solenoid appear in a physical system, it is necessary that some angular variable exists, which is increased in integer number of times on each certain time period, while the phase space is compressed in other directions.

\[ \varphi_{n+1} = M \varphi_n \pmod{2\pi} \]
Chaos generator based on a ring circuit with a non-linear element, driven by a periodic sequence of radio pulses

Model equations

\[ \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \gamma \frac{d}{dt} y g(t), \]

\[ \frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + 4 \omega_0^2 y = \gamma \frac{d}{dt} f(x). \]

\[ f(x) = \frac{x^2}{1 + x^2} \]

\[ g(t) = \begin{cases} 
    a^2 \sin^2 \left( \frac{\pi t}{\tau} \right) \cdot \sin \omega_0 t, & 0 \leq t \leq \tau \\
    0, & \tau \leq t \leq T 
\end{cases} \]
Portrait is visually similar to the Smale-Williams attractor.

The similarity of the image with the iterative diagram of Bernoulli map is clear.

\[ \omega_0 = 6\pi, \tau = 3, T = 13, a = 24, \gamma = 0.4 \]
\[ \Lambda_1 = 0.684 \pm 0.004, \]
\[ \Lambda_2 = -2.11 \pm 0.02, \]
\[ \Lambda_3 = -3.67 \pm 0.04, \]
\[ \Lambda_4 = -5.28 \pm 0.02. \]

Dependence of the Lyapunov exponents for the attractor of the Poincaré map on the parameter \( a \) for fixed values of other parameters

\[ D_{KY} = 1 + \frac{\Lambda_1}{|\Lambda_2|} \approx 1.32 \]

\( \omega_0 = 6\pi, \tau = 3, T = 13, \gamma = 0.4 \)
The distribution of the angles between the stable and unstable manifolds of the attractor of the first system

Chaos generator based on a ring circuit with a nonlinear element and periodically variable band-pass filter

Model equations:

\[
\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = a \gamma \frac{dy}{dt},
\]

\[
\frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + \omega(t)^2 y = a \gamma \frac{d}{dt} f(x).
\]

\[
f(x) = \frac{x^2}{1 + x^2}
\]

\[
\omega(t) = \omega_0 \left(1 + \sin^2 \left(\frac{\pi t}{T}\right)\right)
\]
Portrait is visually similar to the Smale-Williams attractor

The similarity of the image with the iterative diagram of Bernoulli map is clear

ω₀ = 6π, T = 10, a = 20, γ = 0.6
Dependence of the Lyapunov exponents for the attractor of the Poincaré map on the frequency of the first oscillator for fixed values of other parameters $a = 20$, $\gamma = 0.6$

At $\omega_0 = 6\pi$ Lyapunov exponents are

\[
\begin{align*}
\Lambda_1 &= 0.690 \pm 0.002, \\
\Lambda_2 &= -2.20 \pm 0.02, \\
\Lambda_3 &= -4.46 \pm 0.03, \\
\Lambda_4 &= -6.01 \pm 0.02.
\end{align*}
\]

\[
D_{KY} = 1 + \frac{\Lambda_1}{|\Lambda_2|} \approx 1.31
\]
The distribution of the angles between the stable and unstable manifolds of the attractor of the second system
An autonomous system with attractor of Smale–Williams type with resonance transfer of excitation in a ring array of van der Pol oscillators

\[
\begin{align*}
\dot{x}_0 &= \left( A + \frac{3}{2}x_N^2 + x_0^2 - 2S \right) \dot{x}_0 + \omega^2 x_0 = \epsilon x_N^2, \\
\dot{x}_1 &= \left( A + \frac{3}{2}x_0^2 + x_1^2 - 2S \right) \dot{x}_1 + 2^{\frac{2}{N}} \omega^2 x_1 = \epsilon x_0, \\
\dot{x}_2 &= \left( A + \frac{3}{2}x_1^2 + x_2^2 - 2S \right) \dot{x}_2 + 2^{\frac{4}{N}} \omega^2 x_2 = \epsilon x_1, \\
&\vdots \\
\dot{x}_N &= \left( A + \frac{3}{2}x_{N-1}^2 + x_N^2 - 2S \right) \dot{x}_N + 2^{-\frac{2}{N}} \omega^2 x_N = \epsilon x_{N-1}, \\
S &= \sum_{k=0}^{N} x_k^2,
\end{align*}
\]
\( N = 14, \ A = 1, \ \omega_0 = 2\pi, \ \epsilon = 0.2, \ \gamma = 0.1 \)

\[
\dot{z} + \gamma z = x_0^2 + \omega_0^{-2} \dot{x}_0^2 - x_1^2 - 2^{2/N} \omega_0^{-2} \dot{x}_1^2
\]
\[ \lambda_1 = 0.0030479, \]
\[ \lambda_2 = -0.0000065, \]
\[ \lambda_3 = -0.3353, \]
\[ \lambda_4 = -0.6318, \ldots \]

\[ \Lambda_1 = \lambda_1 T_{av} = 0.6945. \]

\[ D_{KY} \approx 2.009. \]
Attractor of Smale-Williams type in a modified Swift-Hohenberg equation

\[ \partial_t u + (1 + \partial_x^2)^2 u = \mu u + u^3 - \frac{1}{5} u v^2 + \varepsilon v \cos 3x, \]
\[ \partial_t v = -v + u^2 v + u^2. \]

boundary conditions
\[ u(x + L, t) = u(x, t), \]
\[ v(x + L, t) = v(x, t). \]


\[ \partial_t u + (1 + \kappa^2(t) \partial_x^2)^2 u = [A + B \chi(x)]u - u^3. \]
Spatio-temporal realizations of variables $u$ и $v$.

$\mu = 0.03$, $\varepsilon = 0.03$, $L = 2\pi$, $\Delta t = 0.001$, $\Delta x = L / 64 \approx 0.098$
The similarity of the image with the iterative diagram of Bernoulli map is clear.

Portrait is visually similar to the Smale-Williams attractor.

\[
S = |U_1| - 1 = 0 \quad \text{surface of Poincaré section}
\]

\(U_1\) is first spatial Fourier-component of \(u\) and \(\phi\) is an argument of \(U_1\).
Lyapunov exponents

\[ \Lambda_1 = 0.665 \approx \ln 2, \]
\[ \Lambda_2 = -42.26, \]
\[ \Lambda_3 = -44.51, \]
\[ \Lambda_4 = -46.46, \ldots \]

Average period 50.37

\[ \mu = 0.03, \, \varepsilon = 0.03, \, L = 2\pi \]

\[ \Delta t = 0.001, \quad \Delta x = \frac{L}{64} \approx 0.098 \]
5D-model

\[
\dot{C}_1 = (\mu - \frac{1}{5} w^2 + 3 |C_1|^2 - \frac{2}{5} |C_2|^2) C_1 - \frac{2}{5} C_2 C_1^* w + \varepsilon C_2^* / 2
\]

\[
\dot{C}_2 = (2 |C_1|^2 - 1) C_2 + (w + 1) C_1^2
\]

\[
\dot{w} = -w + 2 \text{Re}(C_2 C_1^*) + 2 |C_1|^2 (w + 1)
\]

Ansatz

\[
u = C_1 e^{ix} + C_1^* e^{-ix}
\]

\[
\nu = w + C_2 e^{2ix} + C_2^* e^{-2ix}
\]
Portrait of attractor of Poincaré map

Diagram of phase $\varphi = \arg C_1$

cross-section $S = \frac{|C_1|}{2} - 1 = 0$
Lyapunov exponents

\[ \Lambda_1 = 0.65 \]
\[ \Lambda_2 = -46.95 \]
\[ \Lambda_3 = -49.57 \]
\[ \Lambda_4 = -51.14 \]

Average period is 52.61
7D-model

\[ \dot{C}_1 = (\mu - \frac{1}{5} w^2 + 3 \mid C_1 \mid^2 - \frac{2}{5} \mid C_2 \mid^2 - \frac{2}{5} \mid C_4 \mid^2)C_1 - \frac{2}{5} C_2 C_1^* w - \frac{2}{5} C_1^* C_2 C_4 + \frac{1}{2} \varepsilon (C_2^* + C_4) \]
\[ \dot{C}_2 = (2 \mid C_1 \mid^2 - 1)C_2 + C_4 C_1^* + (w + 1)C_1^2 \]
\[ \dot{C}_4 = (2 \mid C_1 \mid^2 - 1)C_4 + C_1^2 C_2 \]
\[ \dot{w} = -w + 2 \text{Re}(C_2 C_1^* ) + 2 \mid C_1 \mid^2 (w + 1) \]

Ansatz

\[ u = C_1 e^{ix} + C_1^* e^{-ix} \]
\[ v = w + C_2 e^{2ix} + C_2^* e^{-2ix} + C_4 e^{4ix} + C_4^* e^{-4ix} \]
Portraits of attractor of Poincaré map

Diagram of phase $\varphi = \arg C_1$

cross-section $S = |C_1|/2 - 1 = 0$
Lyapunov exponents

\[ \Lambda_1 = 0.67 \]
\[ \Lambda_2 = -41.72 \]
\[ \Lambda_3 = -46.34 \]
\[ \Lambda_4 = -47.35 \]
\[ \Lambda_5 = -47.62 \]
\[ \Lambda_6 = -47.81 \]

Average period is 49.6
Histogram for distributions of the angles between the stable and unstable subspaces on the attractor (modified Swift-Hohenberg equation, 5D-model)
Histogram for distributions of the angles between the stable and unstable subspaces on the attractor
(modified Swift-Hohenberg equation, 7D-model)
Smale-Williams attractor in a modified Brusselator model

$$u_t = (A - u)(1 + \varepsilon \cos 6x) - Bu + u^2 v + \gamma(t)\sigma u_{xx},$$

$$v_t = Bu - u^2 v + \gamma(t)v_{xx}.$$ 

Periodic boundary conditions:

$$u(x,t) = u(x + L,t), \quad v(x,t) = v(x + L,t)$$

Diffusion coefficients depend on time:

$$\gamma(t) = \begin{cases} 
1, & nT < t \leq nT + T/2, \\
1/4, & nT + T/2 < t \leq (n+1)T.
\end{cases}$$
Spatio-temporal realizations of variables $u$ and $v$.

$A = 2 \quad B = 4.1 \quad \sigma = 0.25 \quad L = 2\pi \quad T = 32\pi$
Diagram of phase
Of spatial Fourier component of $u$
with wave number $k = 2$.

Portrait of attractor of stroboscopic map
Lyapunov exponents

\[ \Lambda_1 = 0.68 \approx \ln 2, \]
\[ \Lambda_2 = -2.53, \quad \Lambda_3 = -30.71, \]
\[ \Lambda_4 = -31.91, \quad \Lambda_5 = -43.99, \]
\[ \Lambda_6 = -44.03, \quad \Lambda_7 = -65.55, \]
\[ \Lambda_8 = -71.32, \quad \ldots \]

\[ A = 2, \quad B = 4.1, \quad \sigma = 0.25, \quad L = 2\pi, \quad T = 32\pi \]
Ansatz:

\[ u(x, t) = A + x_1 \cos 2x + u_1 \sin 2x + x_2 \cos 4x + u_2 \sin 4x, \]
\[ v(x, t) = B / A + y_1 \cos 2x + v_1 \sin 2x + y_2 \cos 4x + v_2 \sin 4x, \]

**Finite-dimensional model**

\[
\dot{x}_1 = -(B + 1)x_1 - \frac{1}{2} \varepsilon x_2 - 4\sigma x_1 \gamma(t) + F_1 \\
\dot{u}_1 = -(B + 1)u_1 + \frac{1}{2} \varepsilon u_2 - 4\sigma u_1 \gamma(t) + G_1 \\
\dot{x}_2 = -(B + 1)x_2 - \frac{1}{2} \varepsilon x_1 - 16\sigma x_2 \gamma(t) + F_2 \\
\dot{u}_2 = -(B + 1)u_2 + \frac{1}{2} \varepsilon u_1 - 16\sigma u_2 \gamma(t) + G_2 \\
\dot{y}_1 = Bx_1 - 4y_1 \gamma(t) - F_1 \\
\dot{v}_1 = Bu_1 - 4v_1 \gamma(t) - G_1 \\
\dot{y}_2 = Bx_2 - 16y_2 \gamma(t) - F_2 \\
\dot{v}_2 = Bu_2 - 16v_2 \gamma(t) - G_2
\]
\[ F_1 = (2B + \frac{1}{2}u_1v_1 + u_2v_2 + x_2y_2 + \frac{3}{4}x_1y_1)x_1 + (A^2 + \frac{1}{2}x_2^2 + \frac{1}{2}u_2^2 + \frac{1}{4}u_1^2)y_1 + \\
+ A(u_1v_2 + u_2v_1 + x_1y_2 + x_2y_1) + BA^{-1}(u_1u_2 + x_1x_2) \\
G_1 = (2B + \frac{1}{2}x_1y_1 + u_2v_2 + x_2y_2 + \frac{3}{4}u_1v_1)u_1 + (A^2 + \frac{1}{2}x_2^2 + \frac{1}{2}u_2^2 + \frac{1}{4}x_1^2)v_1 + \\
+ A(x_1v_2 + u_2y_1 - u_1y_2 - v_1x_2) + BA^{-1}(x_1u_2 - u_1x_2) \]

\[ F_2 = (2B + u_1v_1 + x_1y_1 + \frac{1}{2}u_2v_2 + \frac{3}{4}x_2y_2)x_2 + (A^2 + \frac{1}{2}x_1^2 + \frac{1}{2}u_1^2 + \frac{1}{4}u_2^2)y_2 + \\
+ A(x_1y_1 - u_1v_1) + \frac{1}{2}BA^{-1}(x_1^2 - u_1^2) \]

\[ G_2 = (2B + u_1v_1 + x_1y_1 + \frac{1}{2}x_2y_2 + \frac{3}{4}u_2v_2)u_2 + (A^2 + \frac{1}{2}x_1^2 + \frac{1}{2}u_1^2 + \frac{1}{4}x_2^2)v_2 + \\
+ A(x_1v_1 + u_1y_1) + BA^{-1}x_1u_1 \]
Portrait of attractor of stroboscopic map

Diagram of phase
\[ \phi = \arctan\left(\frac{u_1}{x_1}\right) \]
Lyapunov exponents

\[
\Lambda_1 = 0.672 \approx \ln 2,
\]
\[
\Lambda_2 = -2.49, \quad \Lambda_3 = -138.14,
\]
\[
\Lambda_4 = -148.25, \quad \Lambda_5 = -351.65,
\]
\[
\Lambda_6 = -362.86, \quad \Lambda_7 = -1263.72,
\]
\[
\Lambda_8 = -1264.64, \quad \ldots
\]
Histogram for distributions of the angles between the stable and unstable subspaces on the attractor (modified Brusselator model, reduced system)
String with parametric excitation of standing wave patterns

\[ \rho(x) \partial_{tt} u = -\left( \alpha + u^2 \right) \partial_t u - \gamma u + G(t) \partial_{xx} u \]

Periodic boundary conditions:

\[ u(x + L, t) = u(x, t) \]

\[ \rho(x) = 1 + \varepsilon \sin mk_0 x \] describes distribution of mass along the string

Tension of the string depends on time:

\[ G(t) = 1 + a \cos^2 \frac{\pi t}{T} \sin 2\omega_0 t + b \sin^2 \frac{\pi t}{T} \sin 2n\omega_0 t \]

if \( m = n - 1 \), \( \Rightarrow \phi' = n\phi + \text{const} \)

if \( m = n + 1 \), \( \Rightarrow \phi' = -n\phi + \text{const} \)
\[ \omega_0 = 2\pi, k_0 = 2\pi, L = 1, a = 0.4, b = 0.2, \varepsilon = 0.2, \alpha = 0.4, \gamma = 0.03 \]

\[ T = 60, n = 3, m = 4 \]
Histogram for distributions of the angles between the stable and unstable subspaces on the attractor (modified string equation, reduced system)