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Plane waves, Lie groups and Feynman integrals

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One-dimensional quantum mechanical harmonic oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2 \cdot m} + \frac{m \cdot \omega^2 \cdot \hat{q}^2}{2} \qquad \hat{p} = -i \cdot \frac{\partial}{\partial x} \qquad \hat{q} = x \qquad [\hat{p}, \hat{q}] = -i$$

Planck constant $\hbar = 1$

Green's function:

$$i \cdot \frac{\partial G}{\partial t} = \hat{H} G \qquad \lim_{t \to 0+} G(x, \xi; t) = \delta(x - \xi)$$

Green's function can be calculated as Feynman integral (Feynman, Hibbs):

$$G(x,\xi;t) = \int_{Q(0)=\xi}^{Q(t)=x} \exp\left[i \cdot \int_0^t \left(P(t) \cdot \dot{Q}(t) - \frac{P^2(t)}{2 \cdot m} - \frac{m \cdot \omega^2 \cdot Q^2(t)}{2} \right) \cdot dt \right] \cdot \prod_t \frac{dP(t) \cdot dQ(t)}{2 \cdot \pi}$$

$$G(x,\xi;t) = \sqrt{\frac{m \cdot \omega}{2 \cdot \pi \cdot i \cdot \sin(\omega \cdot t)}} \cdot \exp\left(i \cdot m \cdot \omega \cdot \frac{(x^2 + \xi^2) \cdot \cos(\omega \cdot t) - 2 \cdot x \cdot \xi}{2 \cdot \sin(\omega \cdot t)} \right)$$

For the moment of time $t = \frac{\pi}{2 \cdot \omega}$ **Green's function is:**

$$G\left(x, \xi, \frac{\pi}{2 \cdot \omega}\right) = \sqrt{\frac{m \cdot \omega}{2 \cdot \pi \cdot i}} \cdot \exp(-i \cdot m \cdot \omega \cdot x \cdot \xi)$$

— kernel of Fourier transform

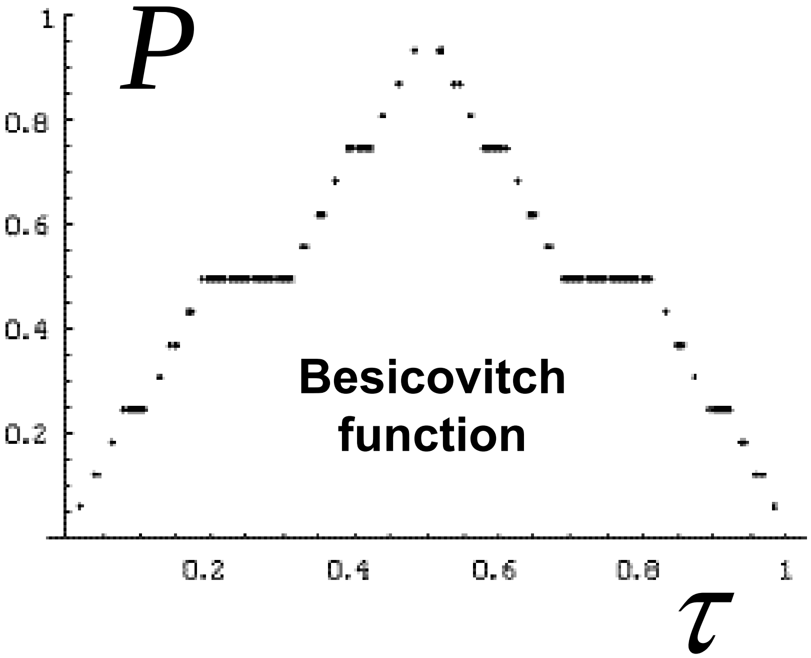
In the system of units m=ω=1 there is the following representation of plane wave by Feynman integral:

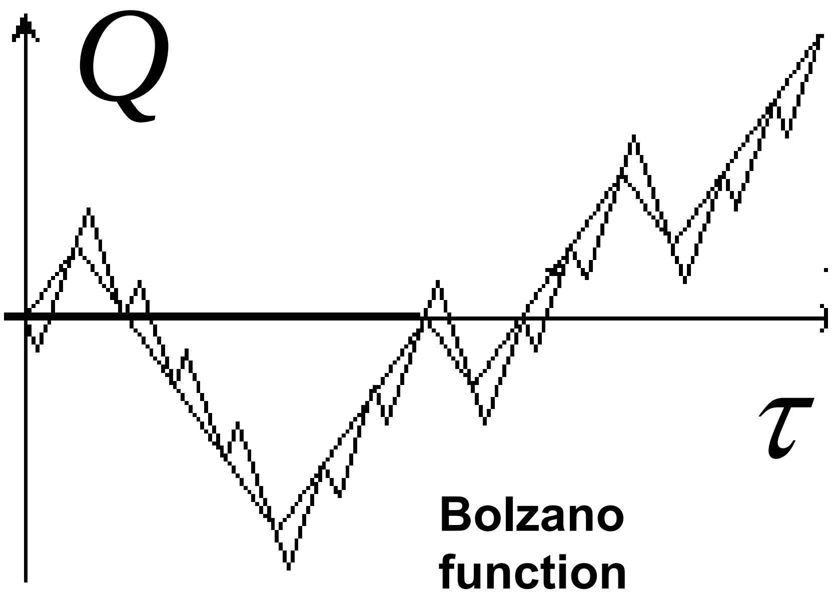
$$\frac{\exp(-i \cdot x \cdot \xi)}{\sqrt{2 \cdot \pi \cdot i}} = \int_{Q(0)=\xi}^{Q(\pi/2)=x} \exp\left[i \cdot \int_0^{\pi/2} \left(P(t) \cdot \dot{Q}(t) - \frac{P^2(t) + Q^2(t)}{2} \right) \cdot dt \right] \cdot \prod_t \frac{dP(t) \cdot dQ(t)}{2 \cdot \pi}$$

Piecewise approximation of phase trajectories in Feynman integral:

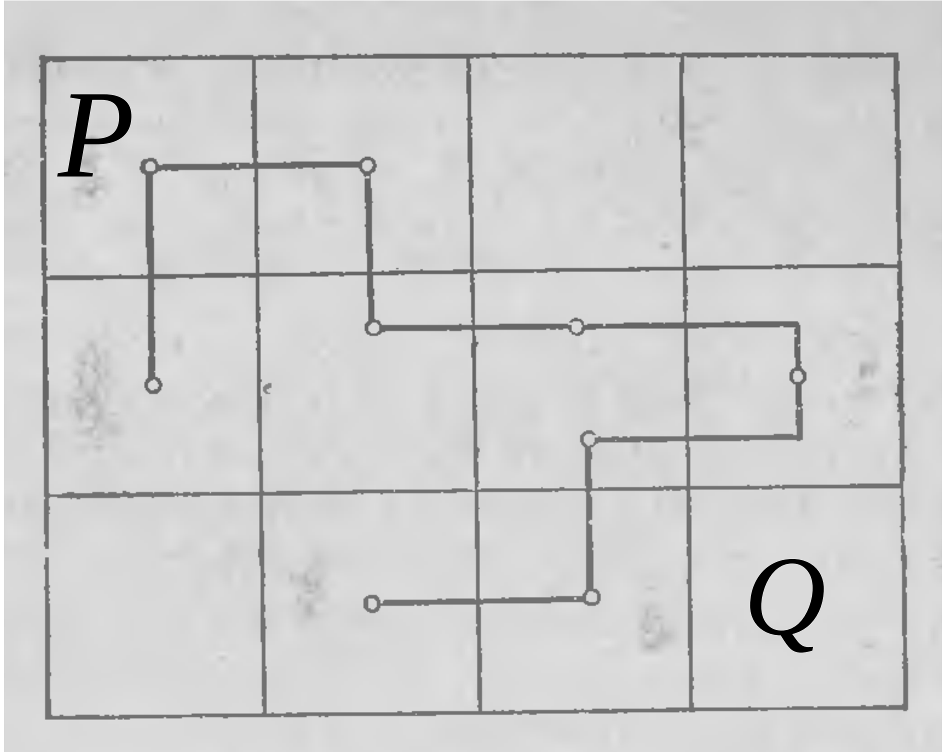
$$Q(\tau) = Q_j + (Q_{j+1} - Q_j) \cdot (\tau - \tau_j) / \Delta\tau, \quad P(\tau) = P_{j+1}, \quad \tau \in [\tau_j, \tau_{j+1}]$$

$$j = \overline{0, N} \qquad \tau_j = j \cdot \Delta\tau \qquad \Delta\tau = \frac{\pi}{2 \cdot (N+1)} \qquad N \to \infty$$



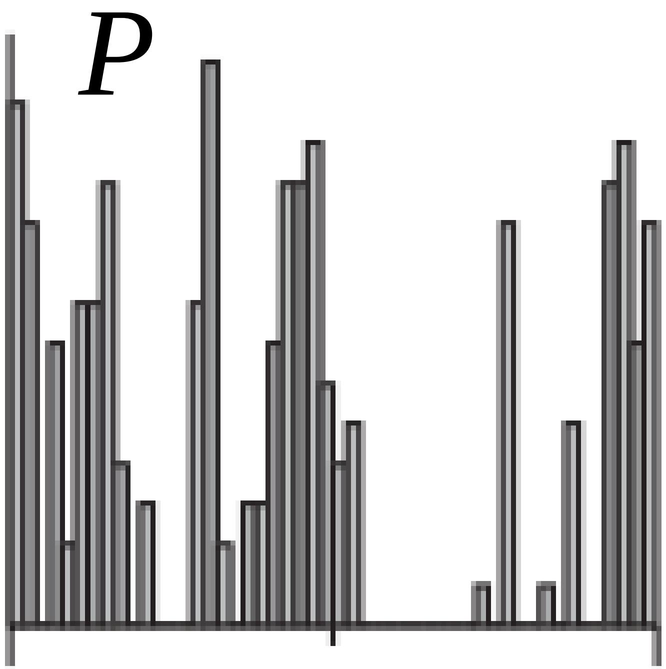


— fractal properties of trajectories in phase space (P,Q)



Two-dimensional random walk on the lattice with decreasing step is subset of phase trajectories on plane (P,Q).

Chaotic behaviour of momenta in phase trajectories of Feynman integrals may take place



$$P_{j+1} = f_1(P_j), \quad P_{j+2} = f_2(P_j, P_{j+1}),$$

$$\ldots,$$

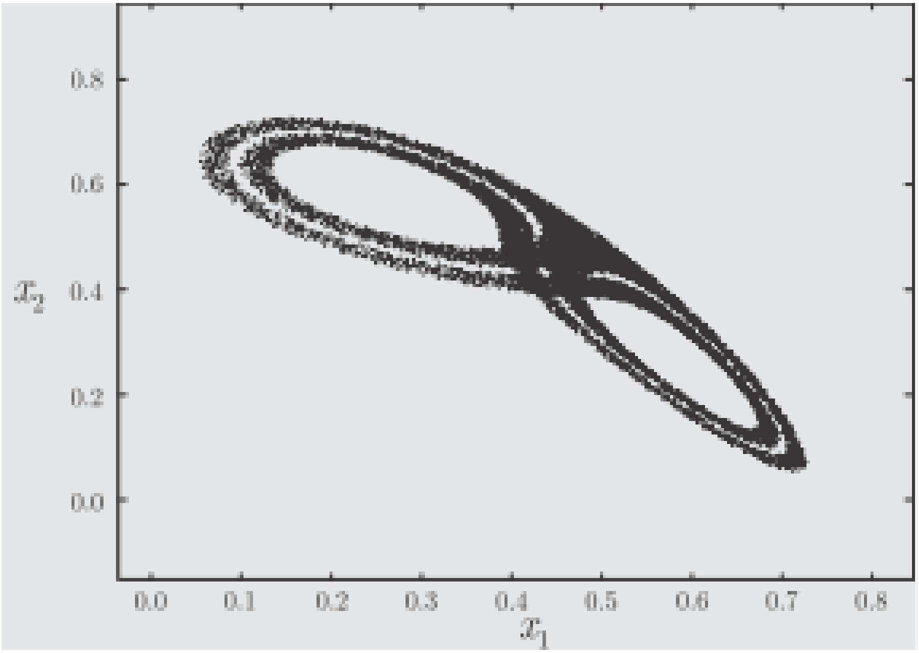
$$P_{j+l} = f_l(P_j, P_{j+1}, \ldots, P_{j+l-1}), \quad \ldots$$

$$P_{j+1} = \pi_j$$

$$\pi_{j+1} = \Pi_j$$

$$\Pi_{j+1} = f_3(P_j, \pi_j, \Pi_j)$$

$$P_{j+3} = f_3(P_j, P_{j+1}, P_{j+2})$$



Gonchenko A.S., Gonchenko S. V., Shilnikov L. P. Towards scenarios of chaos appearance in three-dimensional maps // Rus. J. Nonlin. Dyn., 2012, vol. 8, no. 1, pp. 3–28 (Russian).

$$P_{j+1} = \pi_j \quad \pi_{j+1} = \Pi_j$$

$$\Pi_{j+1} = -0{,}045 + 0{,}85 \cdot \pi_j + 0{,}7 \cdot P_j - \Pi_j^2$$

$$M^3 \times [0,1]$$

Results of V. Z. Grines & C are appropriate too

It is easy to generalize representation of plane wave by Feynman integral for the case of d-dimensional Euclidean space:

$$\frac{\exp\left(-i\cdot\sum_{j=1}^{d}\xi_j\cdot x_j\right)}{(2\cdot\pi\cdot i)^{d/2}}=\int_{\vec{Q}(0)=\vec{\xi}}^{\vec{Q}(\pi/2)=\vec{x}}\exp(i\cdot S[\vec{P},\vec{Q}])\cdot\prod_{j=1}^{d}\prod_{t}\frac{dP_j(t)\cdot dQ_j(t)}{2\cdot\pi}$$

where action is equal to

$$S[\vec{P},\vec{Q}]=\int_0^{\pi/2}\left(\sum_{j=1}^{d}P_j\cdot\dot{Q}_j-\sum_{j=1}^{d}\frac{P_j^2+Q_j^2}{2}\right)\cdot dt$$

$$H(\vec{P},\vec{Q})=\sum_{j=1}^{d}\frac{P_j^2+Q_j^2}{2}$$

Symmetry of this plane wave is closely connected with symmetry of Sp(2d,R)/\H(d) group

Doktorov E. V., Malkin I.A., Man’ko V. I.// J. Mol. Spectrosc. 1975. v. 51. p. 1

Heisenberg-Weyl group H(d):

$$h(\vec{a},\vec{b},z)=\begin{pmatrix}1 & \vec{b}^T & z\\ 0 & E_d & \vec{a}\\ 0 & 0 & 1\end{pmatrix}\qquad \vec{a}=\begin{pmatrix}a_1\\ \dots\\ a_d\end{pmatrix}\qquad \vec{b}=\begin{pmatrix}b_1\\ \dots\\ b_d\end{pmatrix}$$

Symplectic group Sp(2d,R):

$$g^T\cdot\begin{pmatrix}0 & -E_d\\ E_d & 0\end{pmatrix}\cdot g=\begin{pmatrix}0 & -E_d\\ E_d & 0\end{pmatrix}$$

$$\Omega=\sum_{j=1}^{d} dP_j\wedge dQ_j=inv\qquad \frac{d}{dt}\begin{pmatrix}\vec{P}\\ \vec{Q}\end{pmatrix}=\begin{pmatrix}0 & -E_d\\ E_d & 0\end{pmatrix}\cdot\begin{pmatrix}\vec{P}\\ \vec{Q}\end{pmatrix}$$

$$\hat{\vec{P}}=-i\cdot\nabla_d$$ **— operator of momentum**

$$\exp(i\cdot\vec{a}\cdot\hat{\vec{P}})\cdot\exp(-i\cdot\vec{\xi}\cdot\vec{x})=\exp(-i\cdot\vec{\xi}\cdot\vec{a})\cdot\exp(-i\cdot\vec{\xi}\cdot\vec{x})$$

Fractal properties of sets in space of momenta due to application of Williams-Hatchinson theorem:

$$\vec{f}_k : R^d \to R^d$$ — **contraction maps** $k = \overline{1, s}$

$$|\vec{f}_k(\vec{x}) - \vec{f}_k(\vec{y})| \leq \lambda_k \cdot |\vec{x} - \vec{y}| \qquad 0 < \lambda_k < 1$$

$$D_0 \subset R^d$$ — **compact domain**

$$D_{n+1} = \vec{f}_1(D_n) \bigcup \vec{f}_2(D_n) \bigcup \ldots \bigcup \vec{f}_s(D_n)$$

The limit exist:

$$D = \lim_{n \to +\infty} D_n$$

Convergence in Hausdorff metric:

$$\rho_H(U, V) = \max\{\max_{\forall \vec{x} \in U} \min_{\forall \vec{y} \in V} |\vec{x} - \vec{y}|, \quad \max_{\forall \vec{y} \in V} \min_{\forall \vec{x} \in U} |\vec{x} - \vec{y}|\}$$

Case of four-dimensional plane wave: photons

$$\hat{\vec{A}}(\vec{r},t) = \sum_{\vec{k}\alpha} (\hat{c}_{\vec{k}\alpha} \cdot \vec{A}_{\vec{k}\alpha}(\vec{r},t) + \hat{c}^{+}_{\vec{k}\alpha} \cdot \vec{A}^{*}_{\vec{k}\alpha}(\vec{r},t)) \qquad \omega(\vec{k}) = c \cdot |\vec{k}|$$

$$\vec{A}_{\vec{k}\alpha}(\vec{r},t) = \sqrt{4 \cdot \pi} \cdot \frac{\vec{e}^{(\alpha)}(\vec{k})}{\sqrt{2 \cdot \omega(\vec{k})}} \cdot \exp(i \cdot \vec{k} \cdot \vec{r} - i \cdot \omega(\vec{k}) \cdot t) \qquad \vec{k} \cdot \vec{e}^{(\alpha)}(\vec{k}) = 0$$

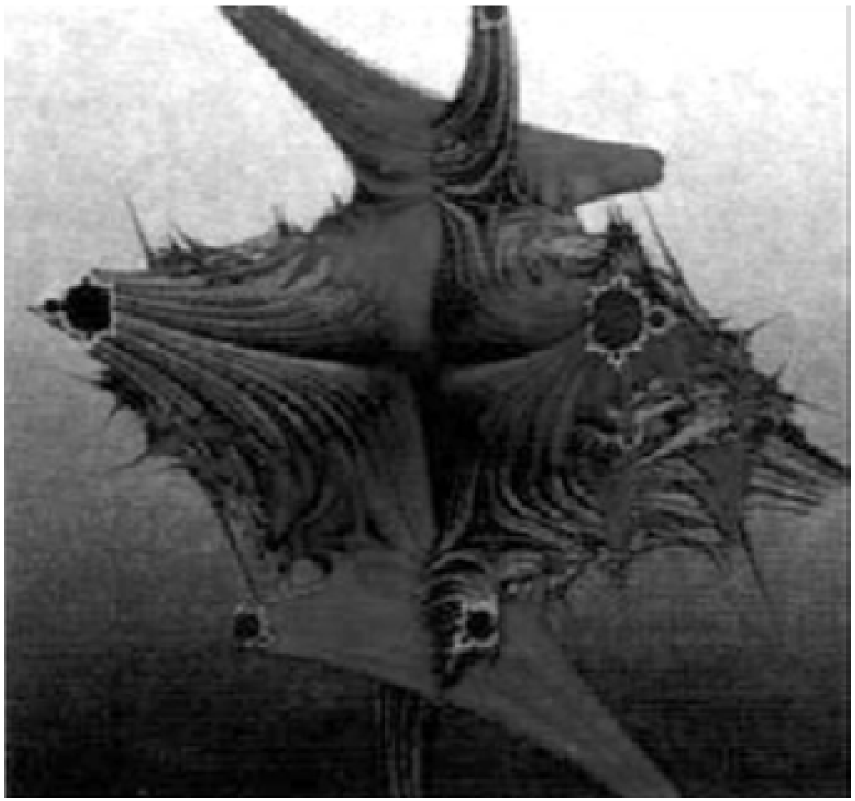
Bose canonical commutative relations:

$$[\hat{c}_{\vec{k}\alpha}, \hat{c}^{+}_{\vec{k}'\alpha'}] = \delta_{\vec{k}\vec{k}'} \delta_{\alpha\alpha'}$$

Eight-dimensional phase space:

$$(P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4)$$

Fractal dynamics of momenta: case of holomorphic succession map



Three-dimensional section of Fatou set

$$z = P_1 + i \cdot P_2 \qquad w = P_3 + i \cdot P_4$$

E. Bedford, D. Smily, 1991

$$\bar{z} = \alpha \cdot z + (\beta \cdot w + z^2)^2$$

$$\bar{w} = \beta \cdot w + z^2$$

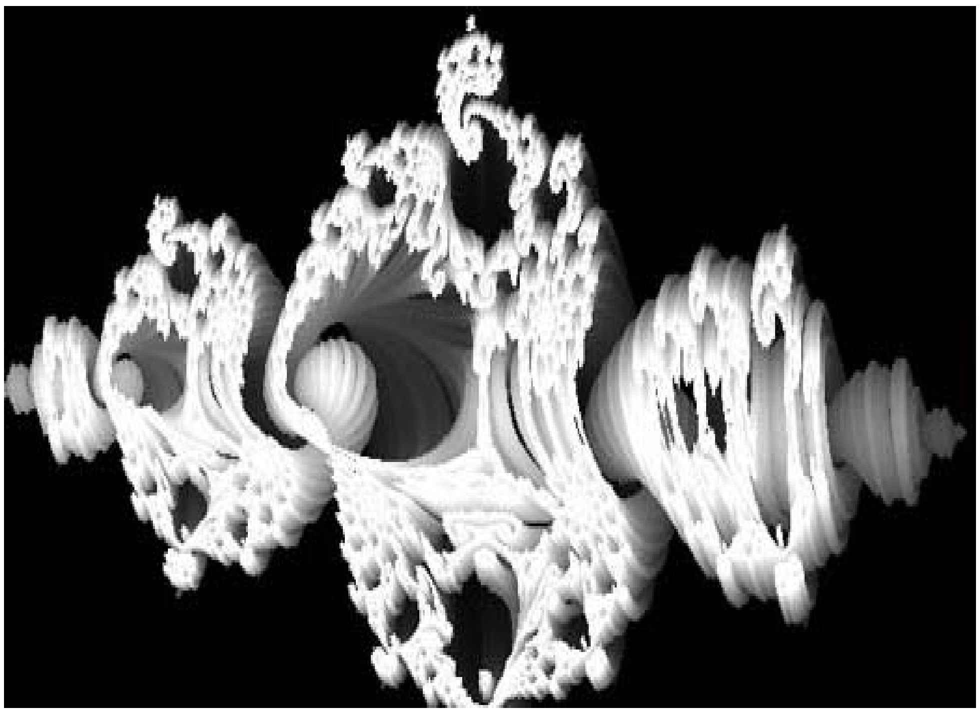
$$\alpha, \beta \in R \quad |\alpha|, |\beta| < 1$$

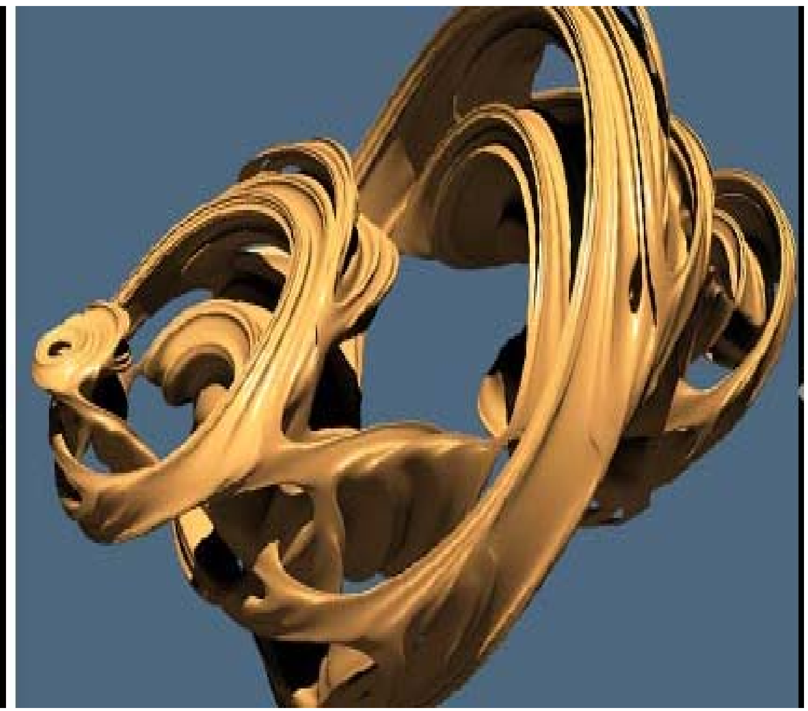
$$(z_0, w_0) \to (0,0)$$

(z_0, w_0) **forms Fatou set**

Mapping of the momenta space into quaternions

$$(P_1, P_2, P_3, P_4) \mapsto P_1 + P_2 \cdot i + P_3 \cdot j + P_4 \cdot k$$





Three imaginary units for quaternions :

$$i^2 = j^2 = k^2 = -1$$

Mapping of the whole eight-dimensional phase space of Feynman integral for plane wave into octonionic algebra

$$(Q_1, Q_2, Q_3, Q_4) \mapsto Q_1 + Q_2 \cdot i + Q_3 \cdot j + Q_4 \cdot k$$

$$x = P + Q \cdot e$$

Additional imaginary units:

$$f = i \cdot e$$

$$g = j \cdot e$$

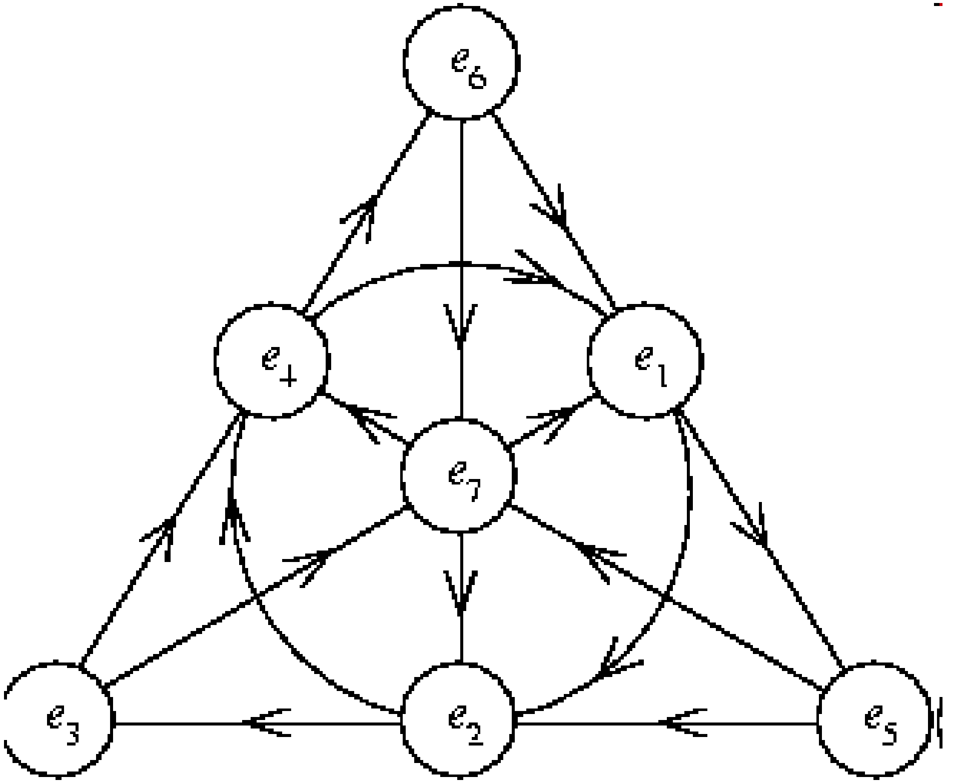
$$h = k \cdot e$$

$$e^2 = -1$$

$$f^2 = -1$$

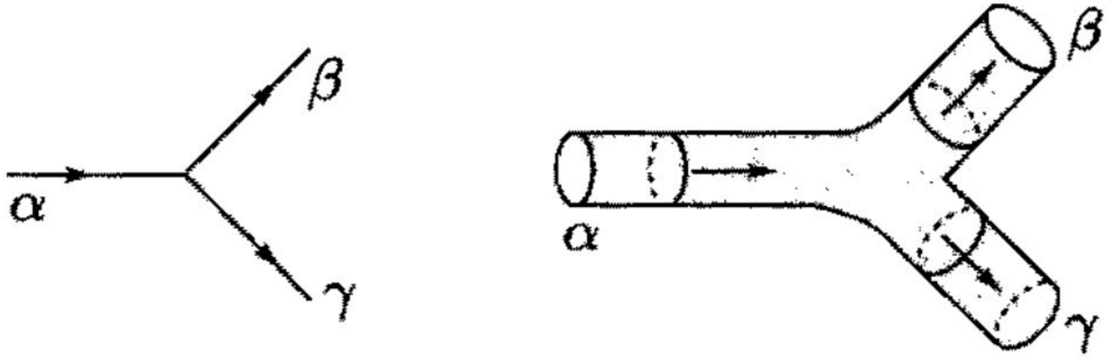
$$g^2 = -1$$

$$h^2 = -1$$



Fano plane

M-theory: all existing theories of strings have uniform dynamic description under D=11



$$X^{\mu}(t,\chi) \in R_D^1$$

$$S[X] = -\tilde{\gamma} \cdot \int_{t_1}^{t_2} dt \cdot \int_0^{\pi} d\chi \cdot \sqrt{\left(\frac{\partial X}{\partial t}\right)^2 \left(\frac{\partial X}{\partial \chi}\right)^2 - \left(\frac{\partial X}{\partial t} \cdot \frac{\partial X}{\partial \chi}\right)^2}$$

Nambu Y. Lectures for the Copenhagen Symposium, 1970

Group G2 is the smallest exceptional simple Lie group

G2 as the subgroup of GL(R7) that preserves the non-degenerate 3-form:

$$dx^{124} + dx^{235} + dx^{346} + dx^{450} +$$
$$+ dx^{561} + dx^{602} + dx^{013}$$

$$dx^{ijk} \equiv dx^i \wedge dx^j \wedge dx^k$$

G2 is the automorphism group of the octonion algebra.

G2 is required in M-theory for compactification of space from D = 11 to D = 4.

Boya L. J. Octonions and M-theory. http://arXiv.org/abs/hep-th/0301037

The main result obtained:

connection of continuous and discrete has been discovered in explicit form

Further developments:

- **investigation of topological entropy of phase trajectories;**
- **investigation of linear functions of momenta in nonMarkovian succession maps.**

References:

- A. A. Potapov, Yu. V. Gulyaev, S. A. Nikitov and alt. The modern methods of image processing. 2008, M., Fizmatlit, 496 p.
- A. A. Potapov, A. E. Rassadin. On Some New Analogies Between Radar-location and Quantum Mechanics // Proc. From XX Int. Scientific-Researh Conf. ‘Radar-location, Navigation, Communication’, Voronezh: NPF “Sakvoee”, 2014, V. 3. pp. 1799-1808.
- A. A. Potapov, A. E. Rassadin. Feynman Integrals, Fractal Radio Systems, or Do We Understand What is Radio Engineering? // Proc From 17th Int. Conf. and Exh. ‘Digital Signal Processing and Its Applications’, Moscow, 2015, V.1, pp. 103-109.
- A. A. Potapov, A. E. Rassadin. On Connection Between Radar-location, Fractal Paradigm and Symmetry of Strong Interactions // Proc. From XXI Int. Scientific-Researh Conf. ‘Radar-location, Navigation, Communication’, Voronezh: NPF “Sakvoee”, 2015, V. 2. pp. 582 - 593.
- A. A. Potapov, A. E. Rassadin. Fractal Paradigm and New Kind of Quantum Quasiparticles in Hydroacoustics // Proc. From IX Int. Scientific Conf. ‘Chaos and Structures in Nonlinear Systems. Theory and Experiment’, Karaganda: E.A. Buketov KarSU Publishing House, 2015, pp. 536-541.
- S. A. Podosjonov, A. A. Potapov, J. Foukzon and alt. Nonholonomic, fractal and connected structures in relativistic continuous media, electrodynamics, quantum mechanics and cosmology / Under ed. of A. A. Potapov. In 3 v. 2015, M., LENAND, 1200 p.
- Potapov A. A., Rassadin A. E. Fractals and path integrals in three-dimensional wave equation // Book of abstracts of Int. Conference-School “Infinite-dimensional dynamics, dissipative systems, and attractors” (Nizhny Novgorod, Russia, July 13-17, 2015) p. 26-27.
- Potapov A. A., Rassadin A. E. Feynmanons in the Korteweg-de Vries equation // Book of abstracts of Int. Conference-School “Infinite-dimensional dynamics, dissipative systems, and attractors” (Nizhny Novgorod, Russia, July 13-17, 2015) p. 28-29.
- A. A. Potapov, A. E. Rassadin . Feynman integrals as connecting component between radio engineering, fractal paradigm and quantum mechanics // Physics of wave processes and radio systems (in print).
- A. A. Potapov, A. E. Rassadin. Feynman integrals, fractal paradigm and new point of view on hydroacoustics // Eurasian Physical Technical journal (in print).
- A. A. Potapov, A. E. Rassadin. On new properties of quasiparticle of radiosystems // Proc. From 25th Int. Crimean Conf. ‘Microwave & Telecommunication Technology’ (in print).