Lyapunov unstable Milnor attractors

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Main definitions

Milnor attractor

For a homeomorphism F of a metric measure space, the *Milnor attractor* is the smallest closed set that contains ω-limit sets of almost all orbits.

Notation: $A_M(F)$ or just A_M.

Important features

- A_M is closed,
- A_M contains any sink,
- $A_M(F) \subset \Omega(F)$.

Main definitions

Milnor attractor

For a homeomorphism F of a metric measure space, the *Milnor attractor* is the smallest closed set that contains ω-limit sets of almost all orbits.

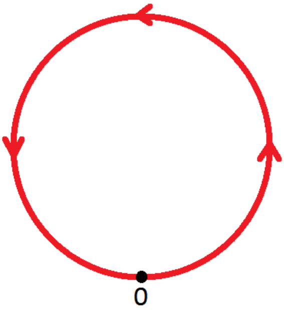
Lyapunov stability

An invariant set K of a homeomorphism F is *Lyapunov stable* provided that for any neighborhood U of K there exists another neighborhood V of K such that any future orbit of F starting at V never quits U.

Genericity

A property is called *topologically generic* if the maps that possess this property form a residual subset in the corresponding space of maps (say, in $\mathrm{Diff}^1(M)$). A property is called *locally topologically generic* if the maps with this property form a residual subset in some open domain.

Milnor attractors can be Lyapunov unstable



$$x \mapsto x + 0.1(1 - \cos x).$$

How typical is this instability?

Problem

Is there an open domain in $\mathrm{Diff}^r(M)$ where all diffeomorphisms have Lyapunov unstable Milnor attractors?

Fact

There exist locally generic diffeomorphisms with Lyapunov unstable Milnor attractors.

Sufficient condition for instability

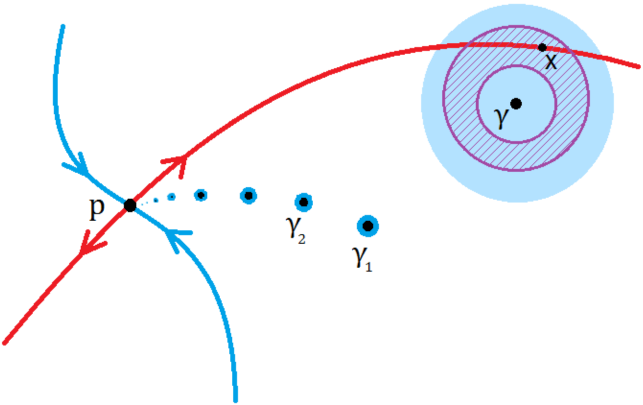
Proposition

Suppose that a diffeomorphism $F \in \mathrm{Diff}^1(M)$ satisfies the following conditions:

- *F has a hyperbolic saddle p whose unstable manifold $W^u(p)$ intersects the basin of attraction of a sink γ.*
- *F has a sequence of periodic sinks $\gamma_j,\ j \in \mathbb{N}$, that accumulate to p, i.e.,*

$$\mathrm{dist}\,(\gamma_j, p) \to 0 \text{ as } j \to \infty.$$

Then $A_M(F)$ is Lyapunov unstable.



The point $x \in W^u(p) \cap B(\gamma)$ is not in A_M, but $\alpha(x) = Orb(p) \subset A_M$. Thus, A_M is Lyapunov unstable. Nevertheless, $\omega(x) = Orb(\gamma) \subset A_M$.

Homoclinic tangencies and instability

Theorem

Suppose that in a dense subset of an open set $U \subset \mathrm{Diff}^r(M), r \geq 1$, *diffeomorphisms exhibit a homoclinic tangency associated with a sectionally dissipative periodic saddle* $p(F)$ *that continuously depends on the map* F *in* U. *Then a topologically generic diffeomorphism in* U *has a Lyapunov unstable Milnor attractor.*

Definitions

Homoclinic tangency

We will say that there is a homoclinic tangency associated with a hyperbolic periodic saddle p if $W^u(Orb(p))$ and $W^s(Orb(p))$ have a point (and therefore an orbit) of non-transverse intersection.

Sectionally dissipative saddle

The saddle p is called *sectionally dissipative*, if it has a unique expanding eigenvalue $\lambda_1 : |\lambda_1| > 1$, and for any two eigenvalues λ_i, λ_j $(i \neq j)$ one has $|\lambda_i \cdot \lambda_j| < 1$.

Proof of the theorem

Proof. (Newhouse argument + Capture lemma).

- Densely in U we have diffeomorphisms with homoclinic tangencies for (the continuation of) a sectionally dissipative saddle.
- A generic unfolding of such a tangency yields a sink that passes close to our saddle.
- (Capture lemma) Unfolding a tangency as above one can also make $W^u(p(F))$ intersect the basin of some sink.
- For $n \in \mathbb{N}$ consider $U_n = \{F \in U \mid F \text{ has a sink } \frac{1}{n}\text{-close to } p(F)\}$. Let $U_0 \subset U$ be the set of diffeos s.t. $W^u(p(F))$ intersects the basin of some sink.
- The sets $U_n,\ n \in \mathbb{N} \cup \{0\}$, are open and dense in U.
- Take $R = \cap U_n$. For any $F \in R$ the hypothesis of the Proposition is satisfied. Apply the Proposition.

□

Corollaries

Corollary

Diffeomorphisms with Lyapunov unstable Milnor attractors are locally topologically generic in $\mathrm{Diff}^r(M)$*, where* $r \geq 2$ *if* $\dim M = 2$ *and* $r \geq 1$ *if* $\dim M \geq 3$.

Corollary

Any C^2*-diffeomorphism* F *exhibiting a homoclinic tangency associated with a sectionally dissipative periodic saddle* p *belongs to the closure of a* C^2*-open set* U *such that a generic diffeomorphism in* U *has an unstable Milnor attractor.*

Corollary

Suppose that in an open set $V \subset \mathrm{Diff}^1(M)$ *a topologically generic diffeomorphism has infinitely many sinks. Then densely in* V *diffeomorphisms have Lyapunov unstable Milnor attractors.*

Definitions

Homoclinic class

The homoclinic class of the saddle p is the closure of the set of transverse homoclinic points of $Orb(p)$:

$$H(p, F) = \overline{W^s(Orb(p)) \pitchfork W^u(Orb(p))}.$$

Dominated splitting

Let Λ be an F-invariant subset of M. A dF-invariant splitting $TM|_\Lambda = E \oplus G$ (with constant fiber dimensions) is called *dominated* if for some $n \in \mathbb{N}$ for any $x \in \Lambda$, and any $u \in E(x), v \in G(x)$ one has

$$\frac{\|dF^n(x)u\|}{\|u\|} \leq \frac{1}{2} \cdot \frac{\|dF^n(x)v\|}{\|v\|}.$$

Dominated splitting or instability

Let M be a closed manifold.

Theorem (C. Bonatti, L. J. Díaz, E. R. Pujals)

For a generic $F \in \mathrm{Diff}^1(M)$ the homoclinic class of any periodic saddle of F either admits a dominated splitting, or is contained in the closure of an infinite set of sinks or sources.

Theorem (I.S.)

For a generic $F \in \mathrm{Diff}^1(M)$

- *either any homoclinic class of F admits some dominated splitting,*
- *or the Milnor attractor is Lyapunov unstable for F or F^{-1}.*

On the definition of attractor

These properties of attractors are generically incompatible:

- A is closed,
- A contains any sink,
- $A(F) \subset \Omega(F)$,
- A is Lyapunov stable.

You can not coin a definition of attractors such that all these conditions hold for your attractors generically.

NB

If you use this slightly different definition of Lyapunov stability:

Definition

An invariant set K is *Lyapunov stable* if for any $\varepsilon > 0$ there is $\delta > 0$ such that any orbit that starts δ-close to K never leaves the ε-neighborhood of K.

then you don't need to require that the attractor should be a closed set. So, actually these three properties are generically incompatible with each other:

- A contains any sink,
- $A(F) \subset \Omega(F)$,
- A is Lyapunov stable.

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