Global stabilisation for damped–driven conservation laws

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20 – 24 July 2015

Outline

Controllability in finite dimension

Consider the control-affine system

$$\dot{u} = V_0(u) + \sum_{j=1}^{m} \zeta^j(t) V_j(u), \quad u(t) \in \mathbb{R}^d, \qquad (1)$$

where $V_0, \ldots, V_m$ are smooth vector fields and ζ^j are scalar integrable functions.

Definition

We say that (1) is *exactly controllable* at time $T > 0$ if for any $u_0, \hat{u} \in \mathbb{R}^d$ there are functions $\zeta^1, \ldots \zeta^m \in L^1(0, T)$ such that the solution of (1) issued from u_0 satisfies

$$u(T) = \hat{u}.$$

This definition is not reasonable in the case of parabolic PDE.

Heat equation with localised control

Example

Consider the heat equation in a bounded domain $D \subset \mathbb{R}^n$:

$$\partial_t u - \Delta u = I_Q(x)\zeta(t,x), \quad x \in D. \tag{2}$$

Here $Q \subset D$ is an open set, I_Q is its indicator function, and ζ is a control function. The heat operator is hypoelliptic:

$$(\partial_t - \Delta)u \in C^\infty(\Omega) \quad \Longrightarrow \quad u \in C^\infty(\Omega).$$

Hence, if $u(t,x)$ is a solution of (2), then $u(T,\cdot) \in C^\infty(D \setminus \overline{Q})$, so that it is easy to find functions $\hat{u}(x)$ that cannot be reached.

Two concepts of controllability for PDE's

Let us consider a nonlinear PDE with an affine control:

$$\partial_t u = F(u) + \zeta(t), \quad u(t) \in H. \tag{3}$$

We assume that for any u_0 in the phase space H Eq. (3) has unique solution $u(t) = S_t(u_0, \zeta)$ issued from u_0. Let $E \subset H$.

Definition

We say that (3) is *approximately controllable* at time T if for any $u_0, \hat{u} \in H$ and any $\varepsilon > 0$ there is $\zeta \in L^\infty(0, T; E)$ such that

$$\|S_T(u_0, \zeta) - \hat{u}\|_H < \varepsilon. \tag{4}$$

We say that (3) is exactly controllable to trajectories at time T if for any trajectory $\hat{u}(t)$ of Eq. (3) with $\zeta \equiv 0$ and any $u_0 \in H$ there is $\zeta \in L^\infty(0, T; E)$ such that

$$S_T(u_0, \zeta) = \hat{u}(T). \tag{5}$$

Formulation of the problem

Let us consider the Burgers equation on the interval $I = [0, \pi]$:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = h(t,x) + \zeta(t,x), \quad x \in I, \tag{6}$$
$$u(t,0) = u(t,\pi) = 0. \tag{7}$$

Here $\nu > 0$ is a parameter, $h \in C^\infty$ is a fixed function, and ζ is a control with range in the space

$$E = \{\zeta \in L^2(I) : \operatorname{supp}\zeta \subset [a,b]\}, \quad 0 \le a < b \le \pi.$$

Due to **Fursikov–Imanuvilov (1995)**, **Diaz (1996)**, **Guerrero–Imanuvilov (2007)** and others, it is known that:

Local exact controllability: *For any solution* $\hat{u}(t,x)$ *of* (6), (7) *with* $\zeta \equiv 0$ *and any* $T > 0$ *there is* $\varepsilon > 0$ *such that, given* $u_0 \in L^2(I)$ *satisfying* $\|u_0 - \hat{u}(0)\|_{L^2} \le \varepsilon$*, one can find a control* $\zeta \in L^2(0,T;E)$ *for which* $u(T) = \hat{u}(T)$*, where* $u(t,x)$ *stands for the solution of* (6), (7) *issued from* u_0*.*

Formulation of the problem

Absence of global approximate controllability: *Suppose that* $[a,b] \neq [0,\pi]$. *Then for any* $T > 0$ *there are initial and target functions* $u_0, \hat{u} \in L^2(I)$ *such that, for any control function* $\zeta \in L^2(0,T;E)$, *we have*

$$\|u(T) - \hat{u}\|_{L^2} \geq 1, \tag{8}$$

where $u(t,x)$ *is the solution of* (6), (7) *issued from* u_0.

Question: *Does the exact controllability to trajectories hold globally? More precisely, given a solution* $\hat{u}(t,x)$ *of* (6), (7) *with* $\zeta \equiv 0$ *and an initial function* $u_0 \in L^2(I)$, *can we find a time* $T > 0$ *and a control* $\zeta \in L^2(0,T;E)$ *such that* $u(T) = \hat{u}(T)$.

Our main result, combined with the Fursikov–Imanuvilov local controllability theory, shows that this property is true.

Global stabilisation of a non-stationary solution

Theorem

Let $\hat{u}(t,x)$ *be a solution of* (6), (7) *with* $\zeta \equiv 0$. *Then for any* $u_0 \in L^2(I)$ *there is* $\zeta \in C^\infty(\mathbb{R}_+ \times I)$ *such that*

$$\zeta(t) \in E \quad \text{for} \quad t \geq 0, \tag{9}$$

$$\|u(t) - \hat{u}(t)\|_{H^1} + \|\zeta(t)\|_{H^1} \leq Ce^{-\alpha t}\|u_0 - \hat{u}(0)\|_{L^1}^{\delta}, \quad t \geq 1, \tag{10}$$

where the positive numbers C, α, *and* δ *do not depend on* u_0 *and* $\hat{u}$.

The proof of this result is based on the following properties:

(a) The L^1 contraction for the difference between two solutions of (6), (7);
(b) The Harnack inequality for the linearised equation;
(c) Comparison principle for solutions of the nonlinear problem.

Global exact controllability to trajectories

The following proposition is a straightforward consequence of the above theorem and the Fursikov–Imanuvilov result on local exact controllability of the Burgers equation.

Proposition

Let $h(t,x)$ be an arbitrary smooth function. Then there is $T > 0$ such that, given any solution $\hat{u}(t,x)$ of problem (6), (7) with $\zeta \equiv 0$ and any initial point $u_0 \in L^2(I)$, one can find a control $\zeta \in L^2(0,T;E)$ such that

$$u(T) = \hat{u}(T).$$

Moreover, the norm of ζ can be bounded by a universal constant depending only on $h(t,x)$ and $\nu > 0$.

Controllability by a low-dimensional localised control

Let us denote by $\mathcal{E}$ the two-dimensional subspace in E spanned by the following functions extended by zero outside $[a, b]$:

$$e_1(x) := \sin\Big(\pi \frac{x-a}{b-a}\Big), \quad e_2(x) = \sin\Big(2\pi \frac{x-a}{b-a}\Big).$$

The proposition below is proved with the help of the result on stabilisation of nonstationary solutions and an adaptation of the Agrachev–Sarychev approach for controlling nonlinear PDE's.

Proposition

Let $h \in C^\infty$ and $\nu > 0$ be fixed. Then for any $\varepsilon > 0$ there is $T > 0$ such that, given any trajectory $\hat{u}(t, x)$ of (6), (7) with $\zeta \equiv 0$ and any $u_0 \in L^2(I)$, one can find $\zeta \in C^\infty(0, T; \mathcal{E})$ for which

$$\|u(T) - \hat{u}(T)\|_{H^1} \leq \varepsilon. \tag{11}$$

Large control space of finite dimension

Let us introduce the spaces

$$\mathcal{E}_N = \mathrm{span}\{e_j, 1 \le j \le N\}, \quad e_j(x) = I_{[a,b]}(x) \sin\Big(j\pi \frac{x-a}{b-a}\Big).$$

It is clear that the union $\cup_N \mathcal{E}_N$ is dense in $E = \{\mathrm{supp}\, \zeta \subset [a,b]\}$. On the other hand, any trajectory $\hat{u}(t)$ of the Burgers equation can be stabilised with the help of an E-valued control $\zeta(t)$:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = h(t,x) + \zeta(t,x), \tag{12}$$

$$\|u(t) - \hat{u}(t)\|_{H^1} \le Ce^{-\alpha t}. \tag{13}$$

By a continuity argument, given $\varepsilon > 0$, one can find a time $T > 0$ and a control $\zeta \in C^\infty(0,T;\mathcal{E}_N)$ such that

$$\|u(T) - \hat{u}(T)\|_{H^1} < \varepsilon. \tag{14}$$

Reduction

Given $\varepsilon > 0$, a finite-dimensional subspace $\mathcal{F} \subset E$, and two functions $u_0, \hat{u} \in H$, we say that (12) is *ε-controllable* with an $\mathcal{F}$-valued control if one can find $\zeta \in C^\infty(0, T; \mathcal{F})$ such that

$$\|u(T) - \hat{u}\|_{H^1} < \varepsilon. \tag{15}$$

The required result will be established if we prove the following:

Proposition

Let $\varepsilon > 0$ and $u_0, \hat{u} \in H$ be fixed. Then, for any integer $k \geq 2$, Equation (12) is ε-controllable with an $\mathcal{E}_k$-valued control if and only if it is ε-controllable with an $\mathcal{E}_{k+1}$-valued control.

Once this property is established, we use ε-controllability of (12) by an $\mathcal{E}_N$-valued control and argue by induction to conclude that (12) is ε-controllable by $\mathcal{E}_2$-valued control. Since $\mathcal{E}_2 = \mathcal{E}$, we obtain the required result.

Extension principle

Let us fix $\varepsilon > 0$, $u_0, \hat{u} \in H$, and $k \geq 2$. Consider the pair of controlled equations

$$\partial_t u + \tfrac{1}{2}\partial_x u^2 - \nu\partial_x^2 u = h(t) + \zeta(t), \qquad (16)$$

$$\partial_t u + \tfrac{1}{2}\partial_x (u + \eta_1(t))^2 - \nu\partial_x^2 (u + \eta_1(t)) = h(t) + \eta_2(t), \qquad (17)$$

where η_1, η_2, ζ are $\mathcal{E}_k$-valued controls.

Lemma

Equation (16) *is* ε*-controllable if and only if so is Equation* (17)*.*

This is a soft result whose proof is based on approximation and oscillation arguments.

Equation (17) is “easier” to control due to the presence of two control functions.

Convexification principle

Under the same hypotheses, consider the pair of equations

$$\partial_t u + \tfrac{1}{2}\partial_x u^2 - \nu\partial_x^2 u = h(t) + \zeta(t), \qquad (18)$$

$$\partial_t u + \tfrac{1}{2}\partial_x (u+\eta_1(t))^2 - \nu\partial_x^2 (u+\eta_1(t)) = h(t) + \eta_2(t), \qquad (19)$$

where $\eta_1, \eta_2 \in \mathcal{E}_k$ and $\zeta \in \mathcal{E}_{k+1}$ are controls.

Lemma

Equation (18) *is* ε*-controllable by an* $\mathcal{E}_{k+1}$*-valued control* ζ *if and only if so is Equation* (17) *with* $\mathcal{E}_k$*-valued controls* η_1, η_2*.*

The key point of the proof of this result is the inclusion

$$\mathrm{co}\{\eta_2 - (u+\eta_1)\partial_x(u+\eta_1) : \eta_1, \eta_2 \in \mathcal{E}_k\} \supset \{\zeta - u\partial_x u : \zeta \in \mathcal{E}_{k+1}\}.$$

Combining the above lemma with the previous one, we see that (18) is ε-controllable with an $\mathcal{E}_k$-valued control if and only if it is ε-controllable with an $\mathcal{E}_{k+1}$-valued control.

Damped–driven conservation laws

Let $D \subset \mathbb{R}^d$ be a bounded domain with C^2-smooth boundary ∂D and let $Q \subset D$ be an open set. Consider the equation

$$\partial_t u - \nu \Delta u + \operatorname{div}_x A(u) = f(t,x), \quad x \in D. \tag{20}$$

Here $u(t,x)$ is an unknown scalar function, $\nu > 0$ is a parameter, $A : \mathbb{R} \to \mathbb{R}^d$ is a smooth function, and f is an external force of the form

$$f(t,x) = h(t,x) + \zeta(t,x), \tag{21}$$

where $h \in L^\infty(\mathbb{R}_+ \times D)$ is a fixed function and ζ is a control supported in Q. Equation (20) is supplemented with the Dirichlet boundary condition

$$u\big|_{\partial D} = 0, \tag{22}$$

and the initial condition

$$u(0,x) = u_0(x). \tag{23}$$

Global stabilisation

Theorem

Suppose there is $m \geq 0$ such that

$$\sup_{v\in\mathbb{R}} \frac{|A'(v)|}{(1+|v|)^m} < \infty. \tag{24}$$

Then for any $\sigma \in (0,1)$ there are positive numbers C and $\alpha < 1$ such that the following property holds:

Exponential stabilisation. Given $u_0, v_0 \in L^\infty(D)$ one can find a control $\zeta \in C^{\frac{\sigma}{2},\sigma}(\mathbb{R}_+ \times D)$ such that $\operatorname{supp}\zeta \subset \mathbb{R}_+ \times Q$ and

$$\zeta(t) = 0 \quad \text{for } t \leq T := C\log(1 + \|u_0\|_{L^\infty} + \|v_0\|_{L^\infty}),$$

$$\|u(t) - v(t)\|_{C^\sigma} + \|\zeta\|_{C^{\frac{\sigma}{2},\sigma}(D_t^{t+1})} \leq Ce^{-\alpha(t-T)}\|v_0 - u_0\|_{L^1}^{\alpha},$$

where $t \geq T$ and $D_t^{t+1} = [t, t+1] \times D$ in the second inequality.