New invariants of topological conjugacy of non-invertible inner mappings.

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Dedicated to the memory of V. V. Sharko (1949 – 2014)

Topological conjugacy is where topology meets dynamics.

Topological conjugacy preserve trajectories,

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{f} & x & \xrightarrow{f} & f(x) & \xrightarrow{f} & f^2(x) & \xrightarrow{f} & f^3(x) & \xrightarrow{f} & \cdots \\
 & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \\
\cdots & \xrightarrow{g} & y & \xrightarrow{g} & g(y) & \xrightarrow{g} & g^2(y) & \xrightarrow{g} & g^3(y) & \xrightarrow{g} & \cdots
\end{array}$$

thus invariants and classification up to topological conjugacy is the qualitative description of the dynamical system.

Topological theory of dynamics of invertible maps (homeomorphisms and diffeomorphisms) is well developed.

Properties of generic maps and structurally stable maps are described, some classes of maps are totally classified up to the topological conjugacy.

Problem:

Do the same, but for some non-invertible maps.

Obstacle:

Not enough tools
(Not enough known invariant sets of topological conjugacy to distinguish topologically different maps)

Solution:

Introduce lots of new invariants.

Prove their properties and use them for the problem of topological classification of non-invertible inner maps up to conjugacy.

The class of maps considered: (non-invertible) inner mappings.

Some definitions.

Disconnected spaces

Hereditarily disconnected space $\subset$ totally disconnected space $\subset$ 0-dimensional space $\subset$ discrete space

Maps

map	**preimage of any point**
light map	totally disconnected set
0-dimensional map	0-dimensional set
isolated map	discrete (isolated) set
finite-to-one map	finite set

A map $f : M \to M$ is called *open* if the image of an open set is open.

A simple example of a continuous but not open map: a generic cubic polynomial on $\mathbb{R}$.

A continuous map $f : M \to M$ is called *innner mapping* if

1. f is an open map.

2 f is an isolated map.

(Trokhimchuk inner mappings).

Inner mappings are introduced by Stoilov. 1. f is an open map.

2.1 f is an 0-dimensional map. (Stoilov inner mappings).

Characterization of inner mappings on 2-manifolds:

Stoilov theorem. An inner mapping $f = g \circ h$, where g is an analytic function and h is a homeomorphism.

i.e. inner mappings on 2-manifolds are branched coverings.

Why innner maps?

General non-invertible maps are hard to study:
Bad image, bad pre-image.

Whereas innner maps are good.

The class of innner mappings is wide enough.

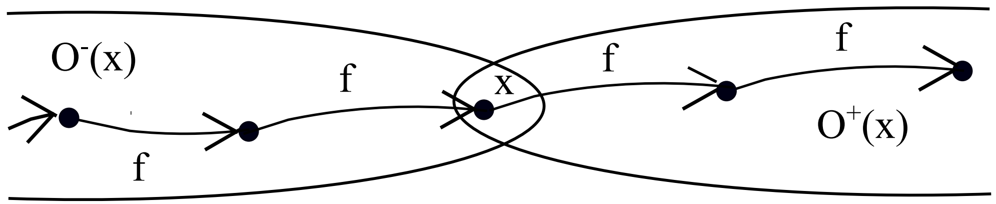
Innner mappings include: homeomorphisms, diffeomorphisms, holomorphic maps, some of the polynomial maps.

It allows unified view, unified theory.

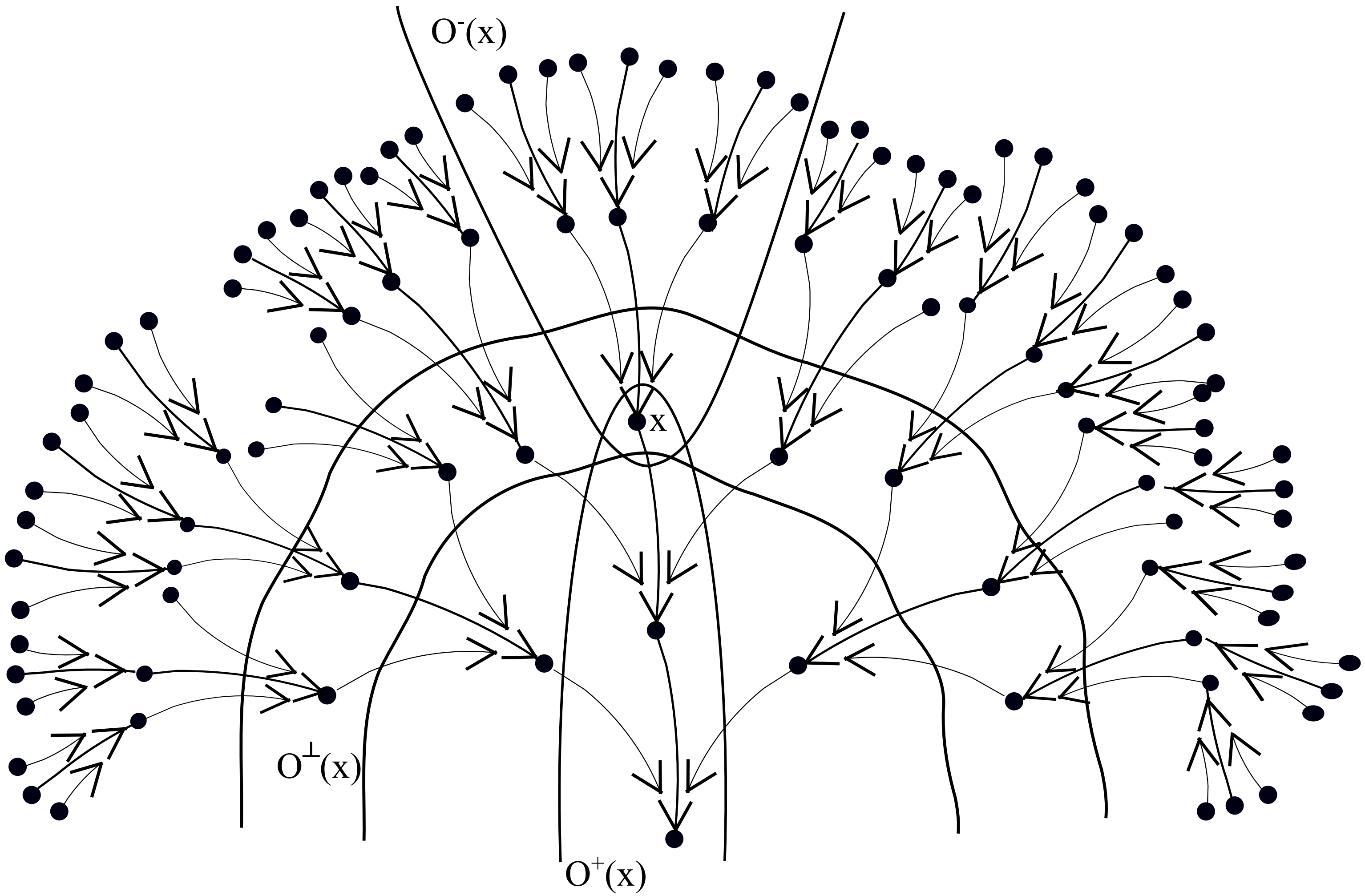
And unified theory creates a way to naturally extend and generalize results obtained for homeomorphisms into results for innner mappings.

And allows some unusual view on holomorphic dynamics.

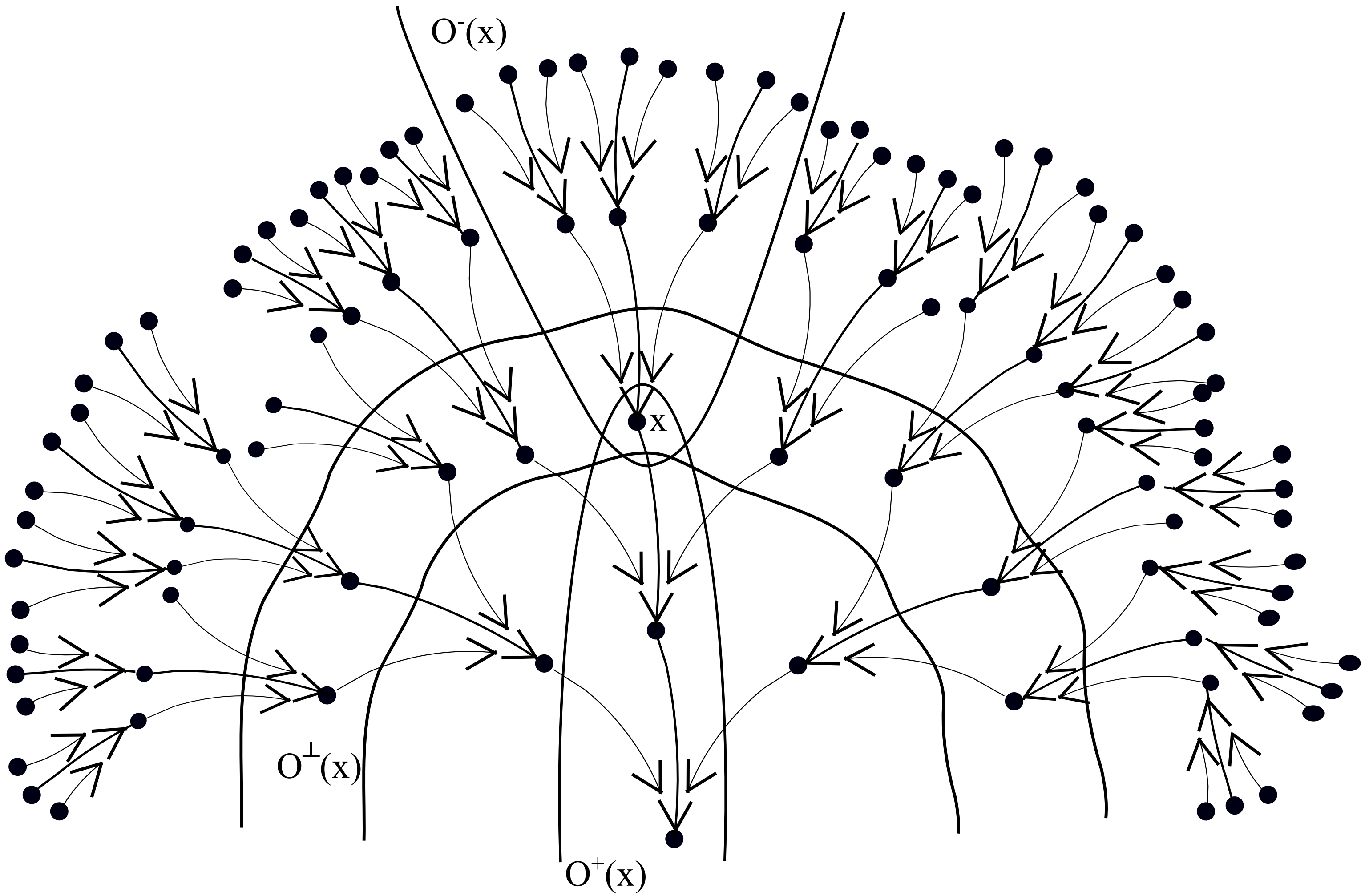
The difference between homeomorphisms and inner mappings.



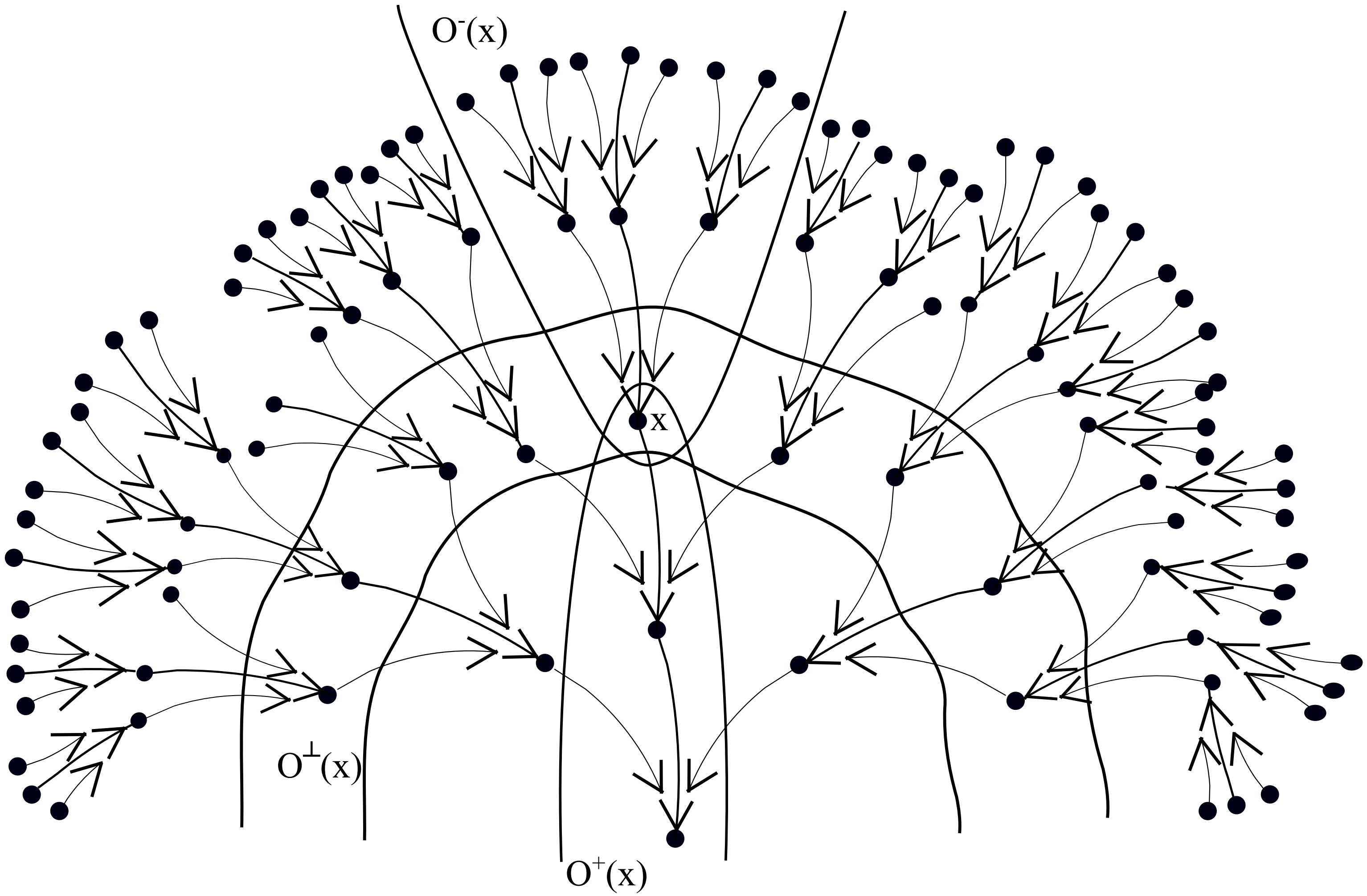
homeomorphism



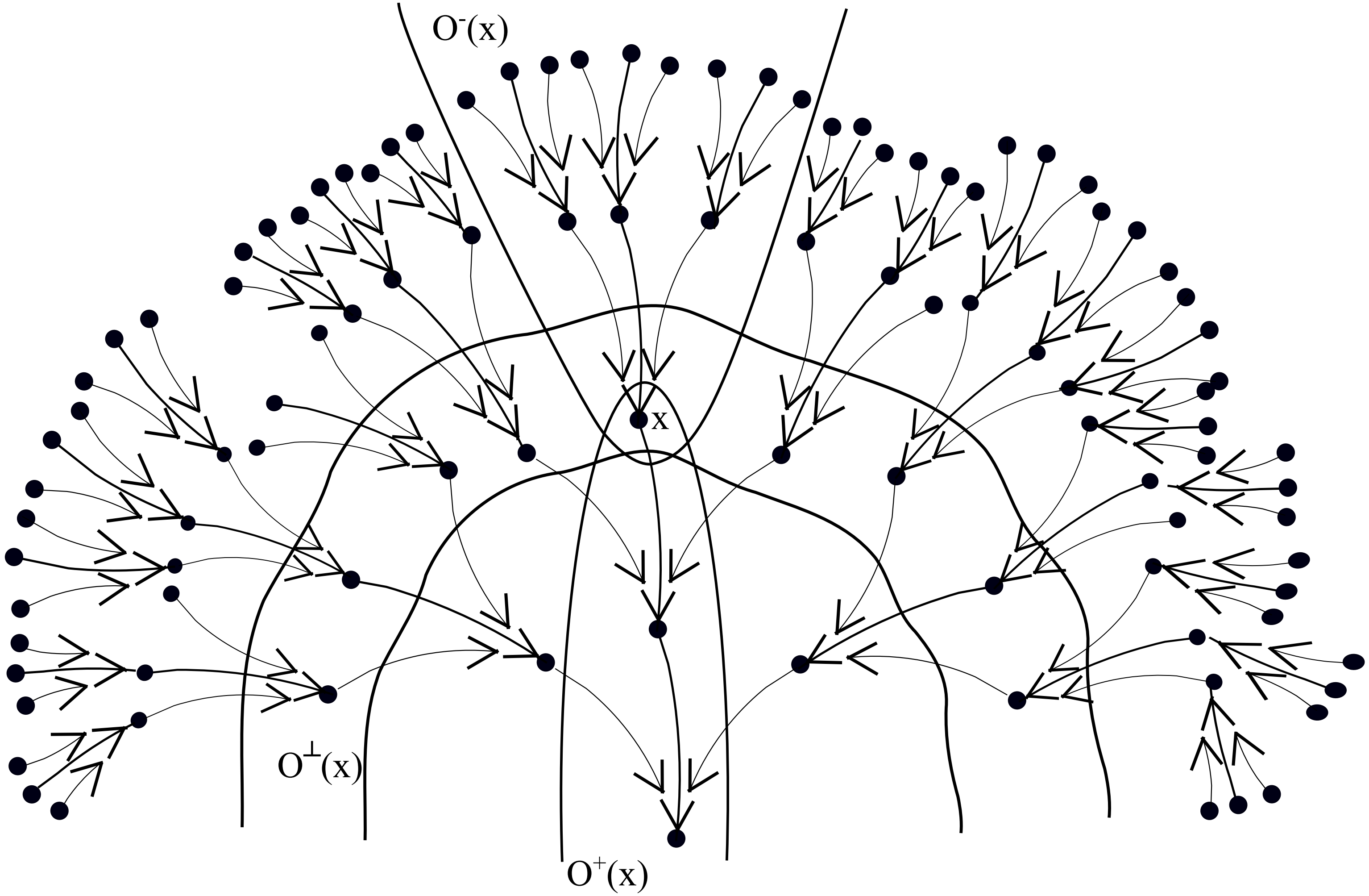
$$O_f^+(x) = \{f^n(x) |\ n \geq 0\}.$$



$$O_f^-(x) = \{f^{-n}(x) |\ n \geq 0\}.$$



$$O_f(x) = \cup_{l,m \geq 0} f^{-l}(f^m(x)).$$



Neutral section $O^{\perp}_{f}(x) = \cup_{n\geq 0} f^{-n}(f^{n}(x))$.

Homeomorphism: 2 directions.

ω - limit set, ω - recurrent points; ω - linked points, ω - non-wandering points.

α - limit set, α - recurrent points; α - linked points, α - non-wandering points.

Inner mappings: ∞ directions, but 2 dimensions.

First dimension - time:

ω - limit set, ω - recurrent points; ω - linked points, ω - non-wandering points.

α - limit set, α - recurrent points; α - linked points, α - non-wandering points.

Inner mappings: ∞ directions, but 2 dimensions.

Second dimension - “neutral” dimension:

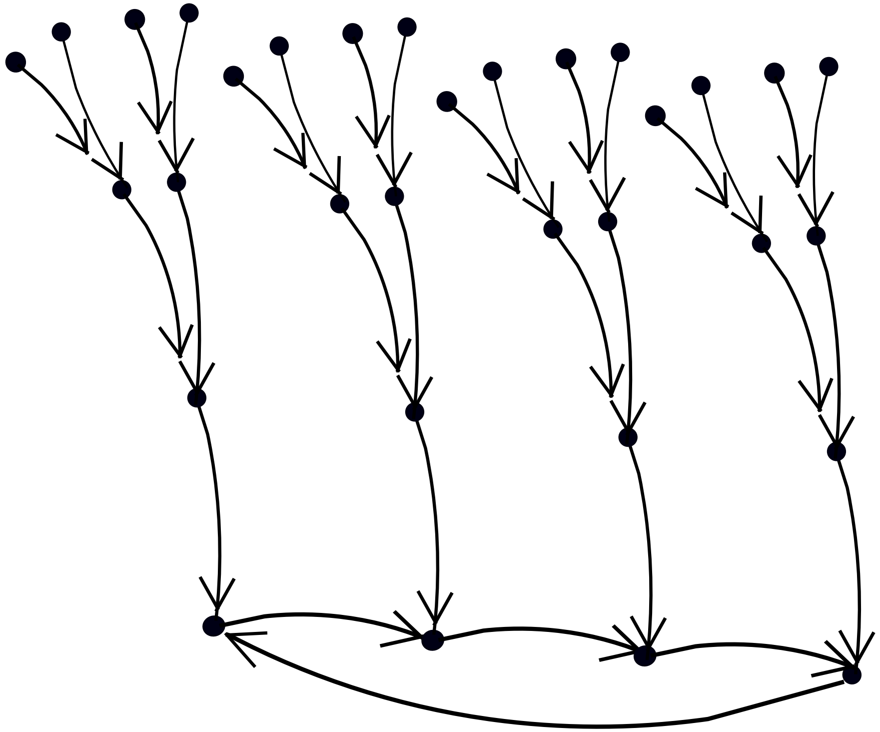
$\perp$ - limit set, $\perp$ - recurrent points; $\perp$ - linked points, $\perp$ - non-wandering points.

Inner mappings: ∞ directions, but 2 dimensions.

Two dimensions allow decomposition.

For example, superwandering points:

$\alpha + \omega$ wandering points AND $\perp$ - wandering points.



Periodic point.

Only part of the wide trajectory belong to the non-wandering set.

$\perp$ - limit set (Neutral limit set)

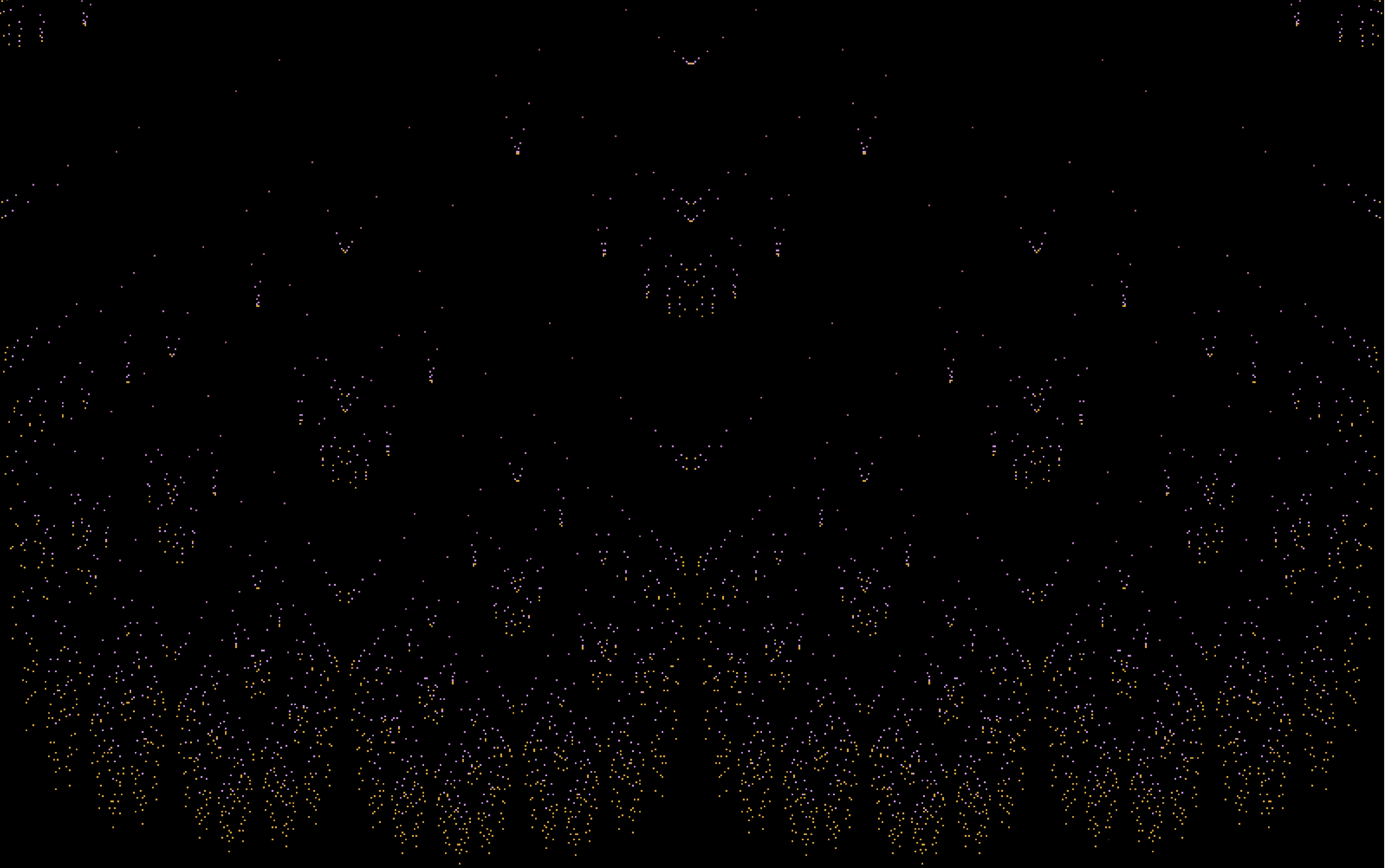
$$\perp(x) = \bigcap_{n \in \mathbb{N}} \overline{O^{\perp}(x) \setminus f^{-n} \circ f^{n}(x)}$$

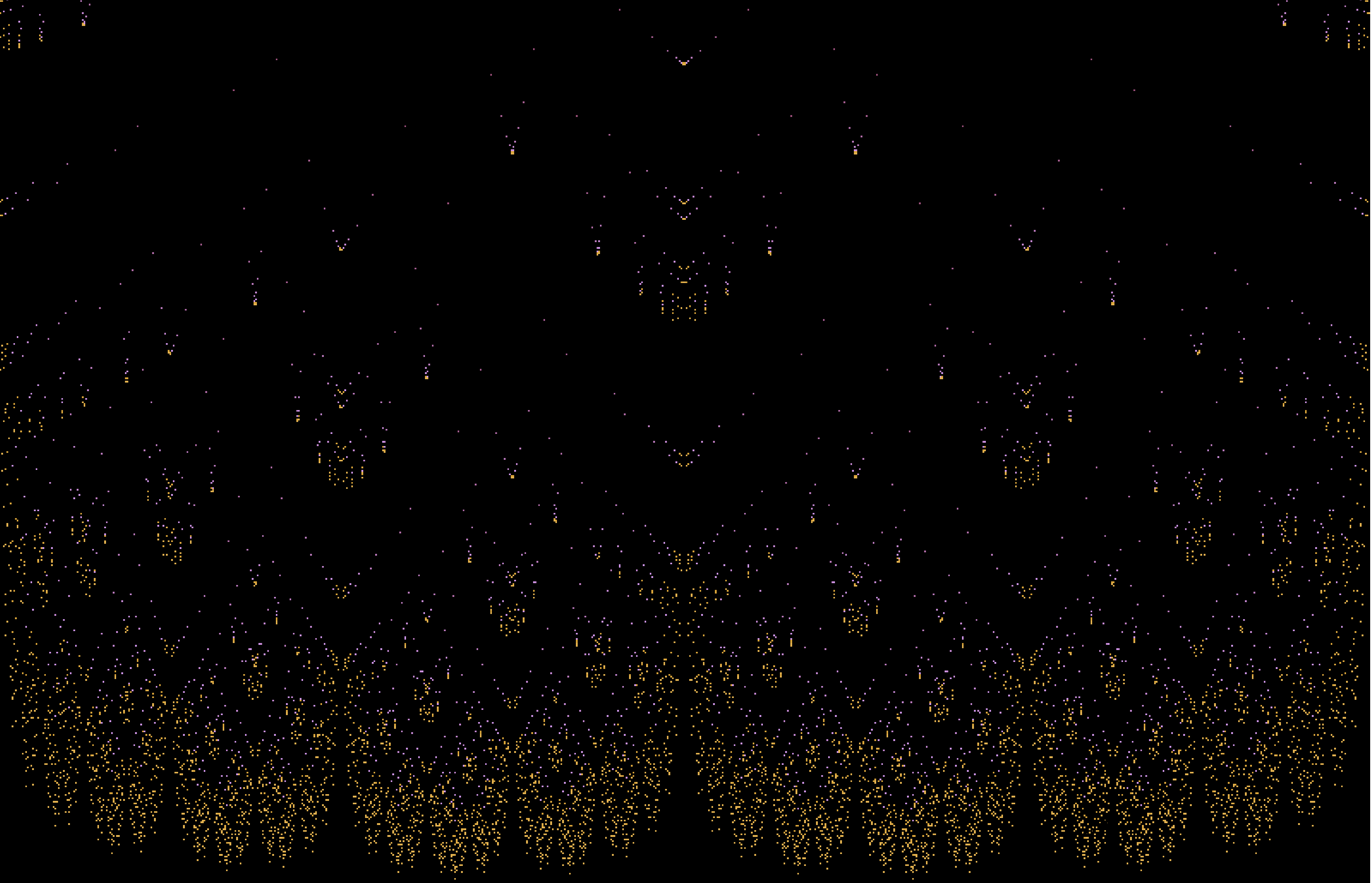
An example:

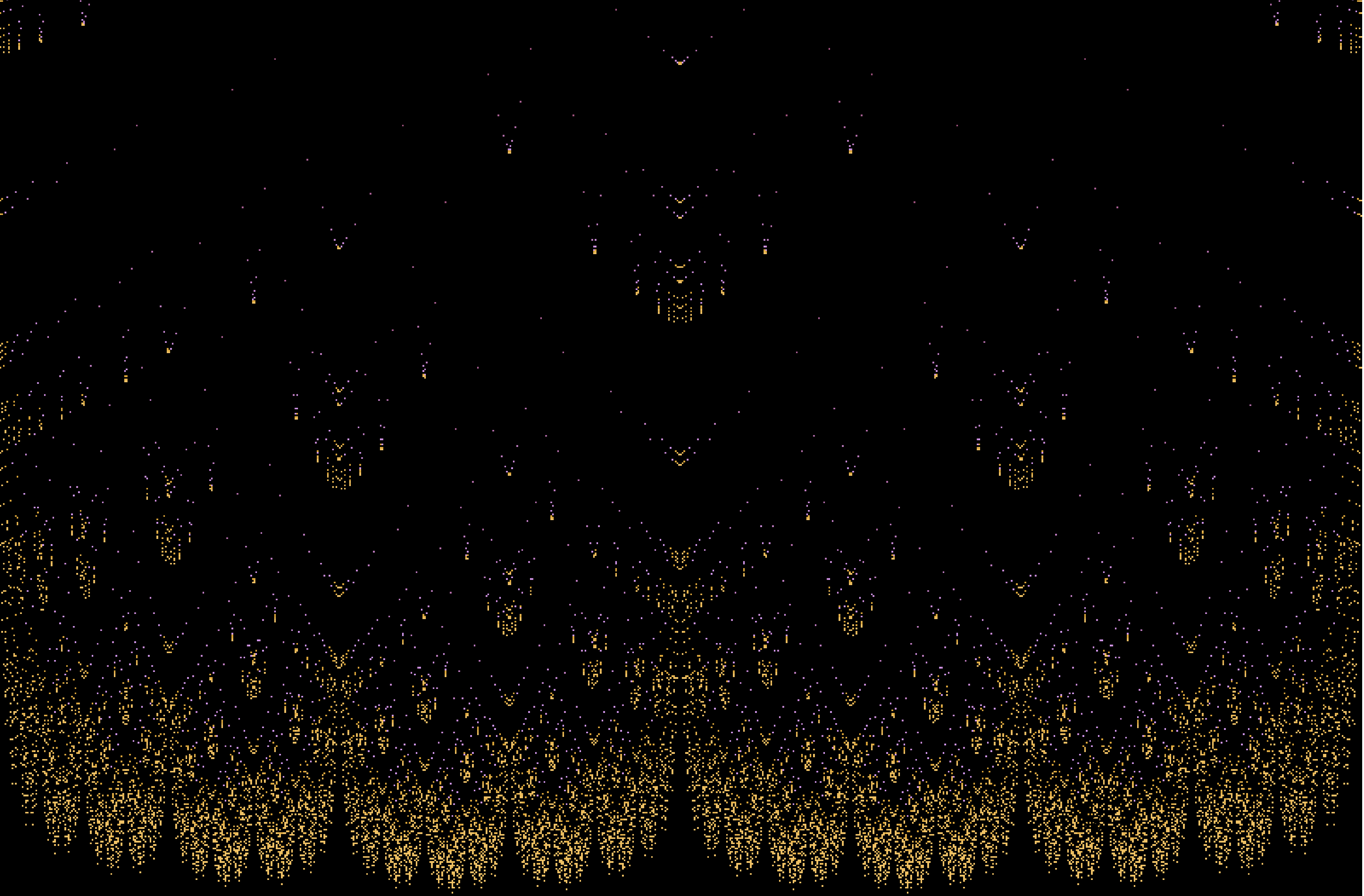
The Neutral limit set for the point (0.93,0) of the map

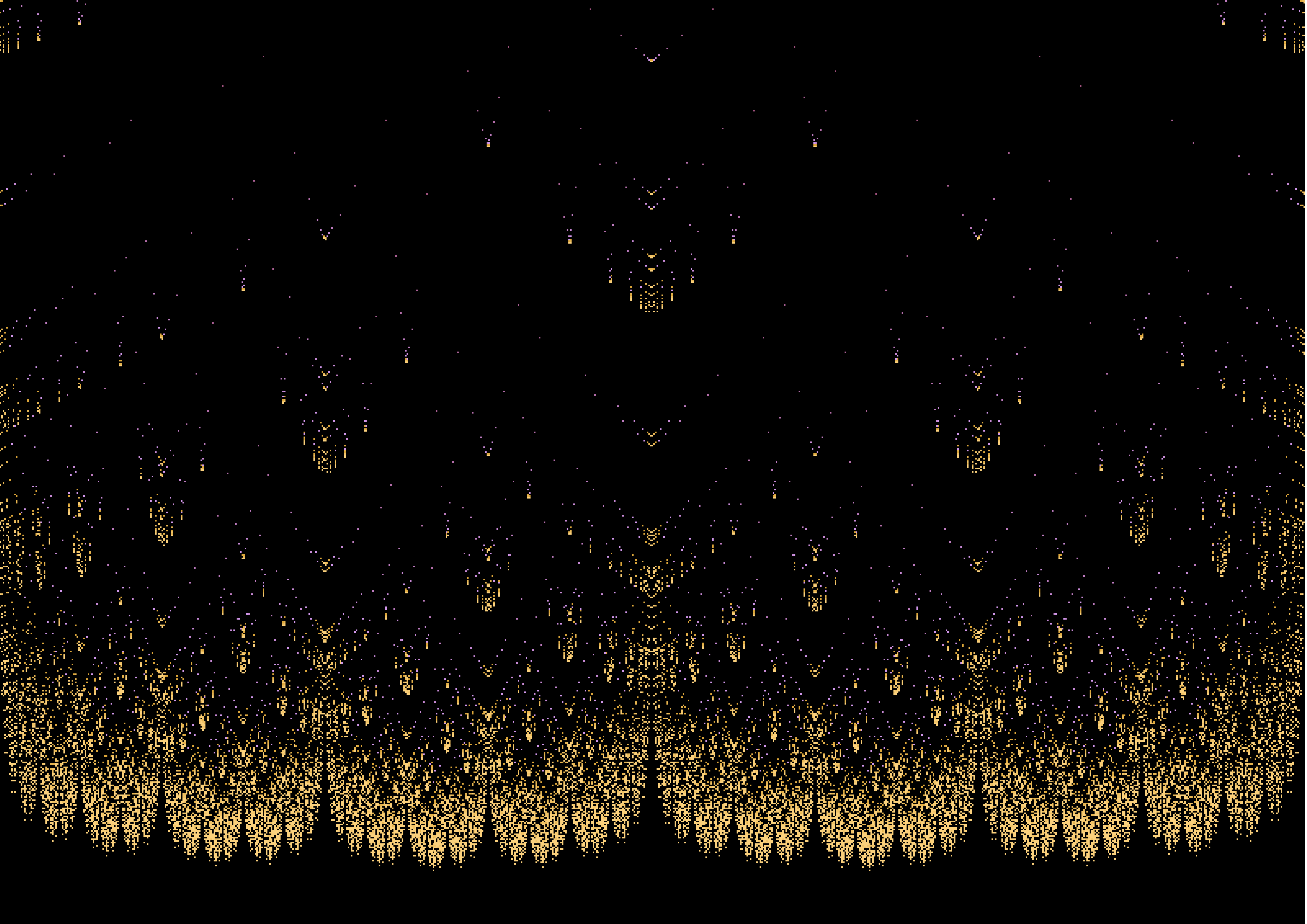
$$(r, \varphi) \mapsto (r + 1 + \frac{1}{5} \sin \pi\varphi \sin^2 \pi r, 2\varphi)$$

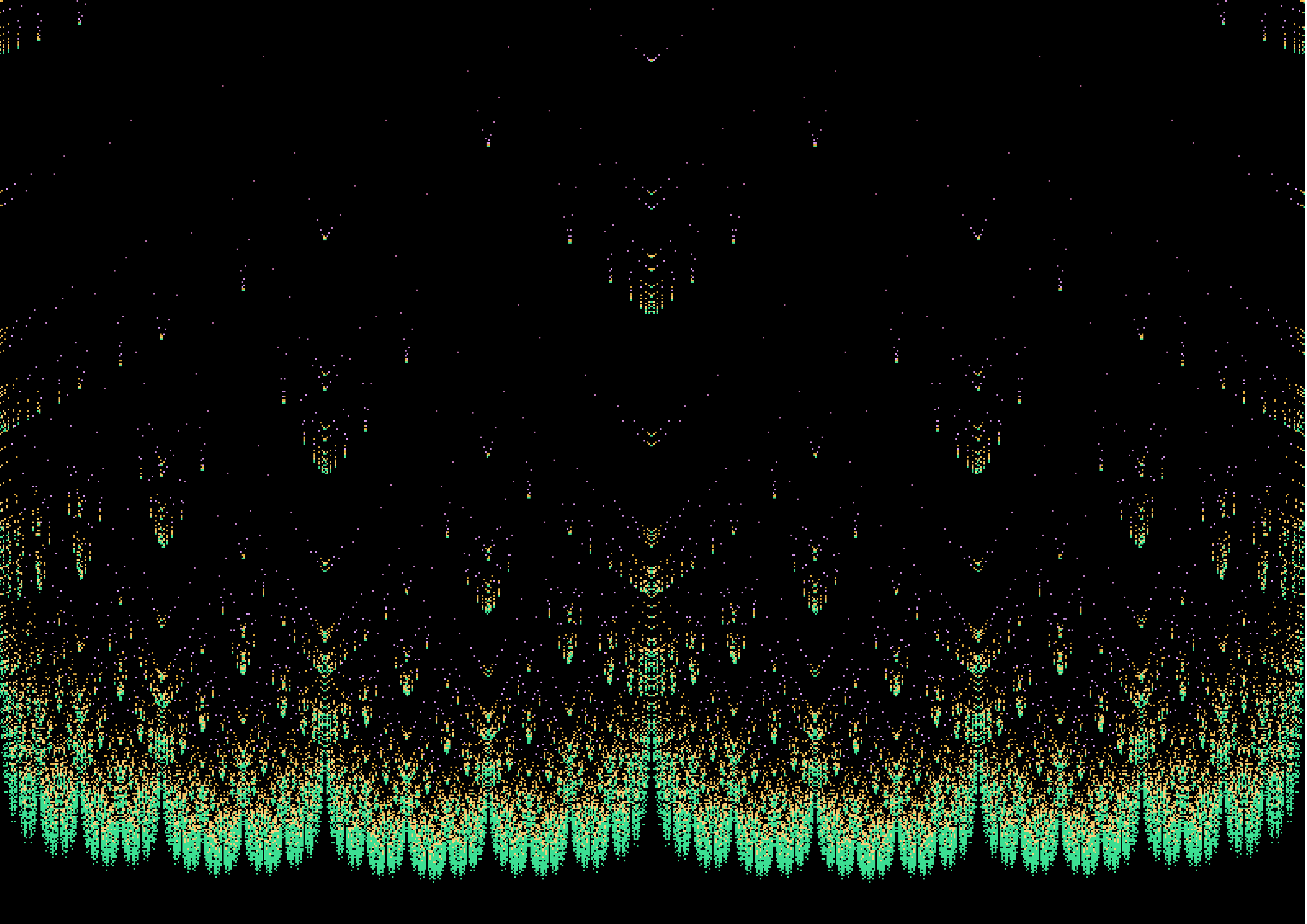
of the cylinder $\mathbb{R} \times S^1$

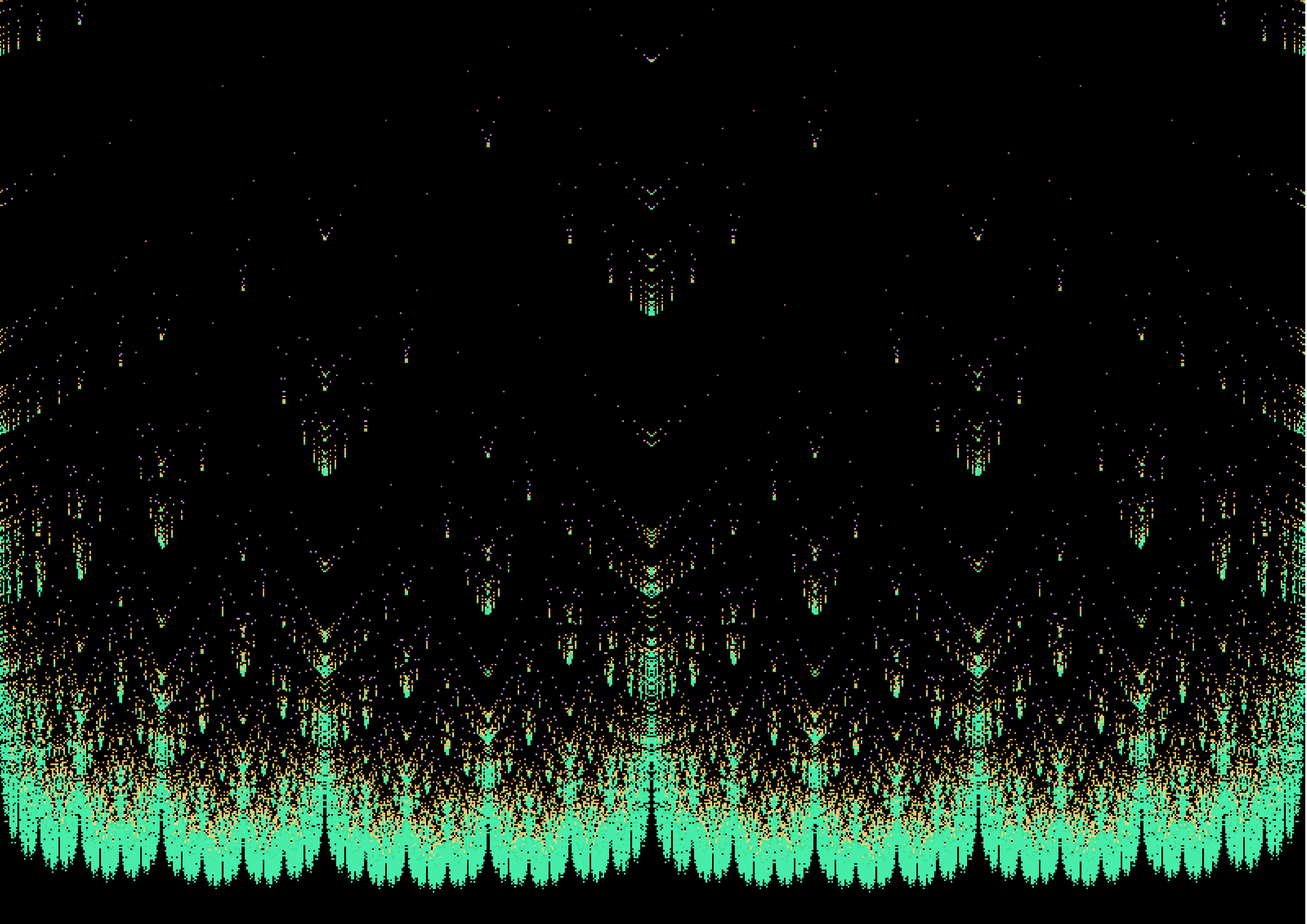


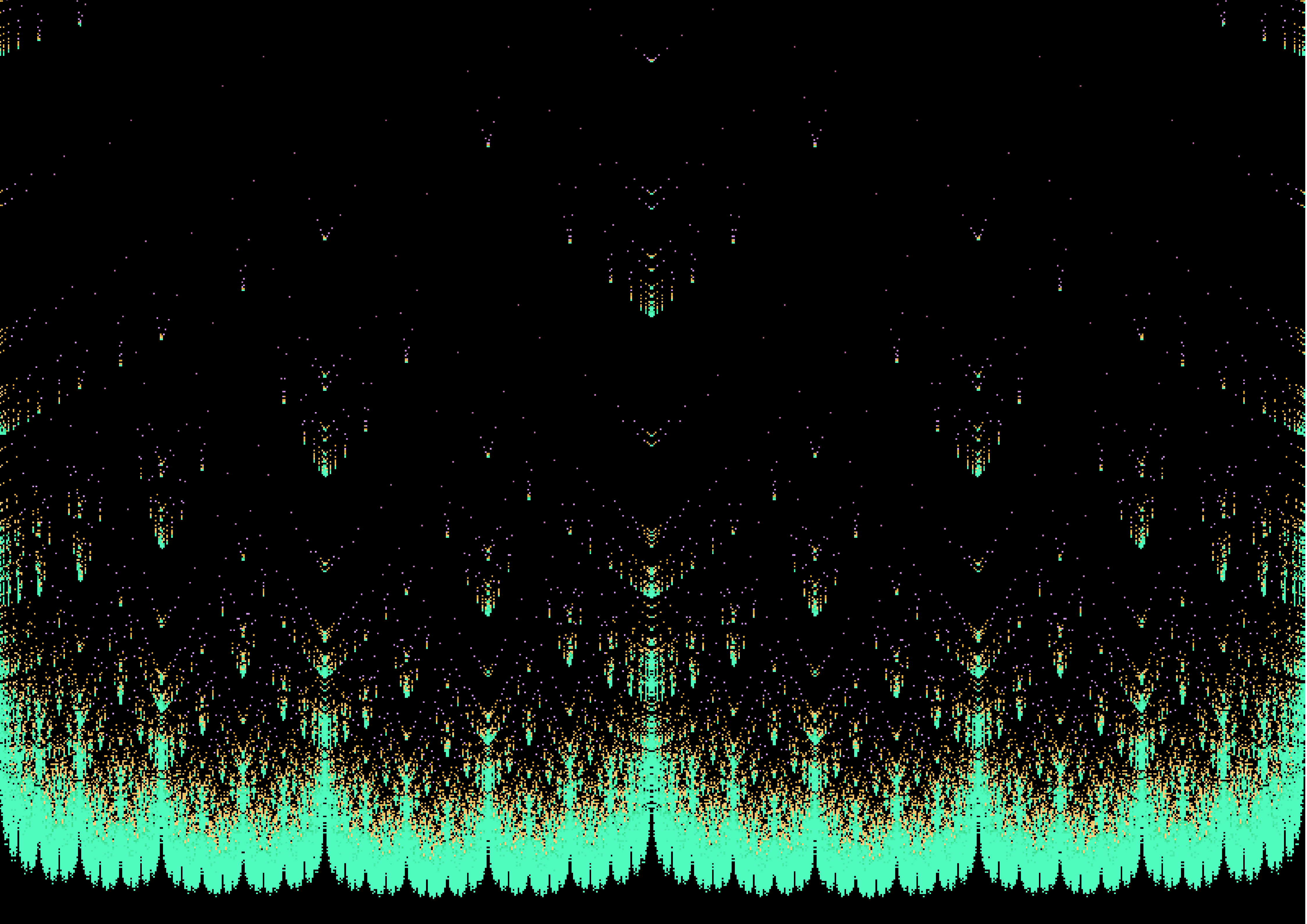












An example of chaotic behaviour:

The Neutral limit set for the point (0.5,0) of the map

$$(r,\varphi) \mapsto (r + 1 + \frac{1}{5}\sin 2\pi\varphi \sin^2 \pi r, 2\varphi)$$

of the cylinder $\mathbb{R} \times S^1$

Neutral component of x

$\mathbf{Z}^{\perp}(x) = \cap_{\varphi} \varphi^{-1}(0)$

$\varphi(x) = 0$, $\varphi(x)$ is constant on each $O_f^{\perp}(x)$

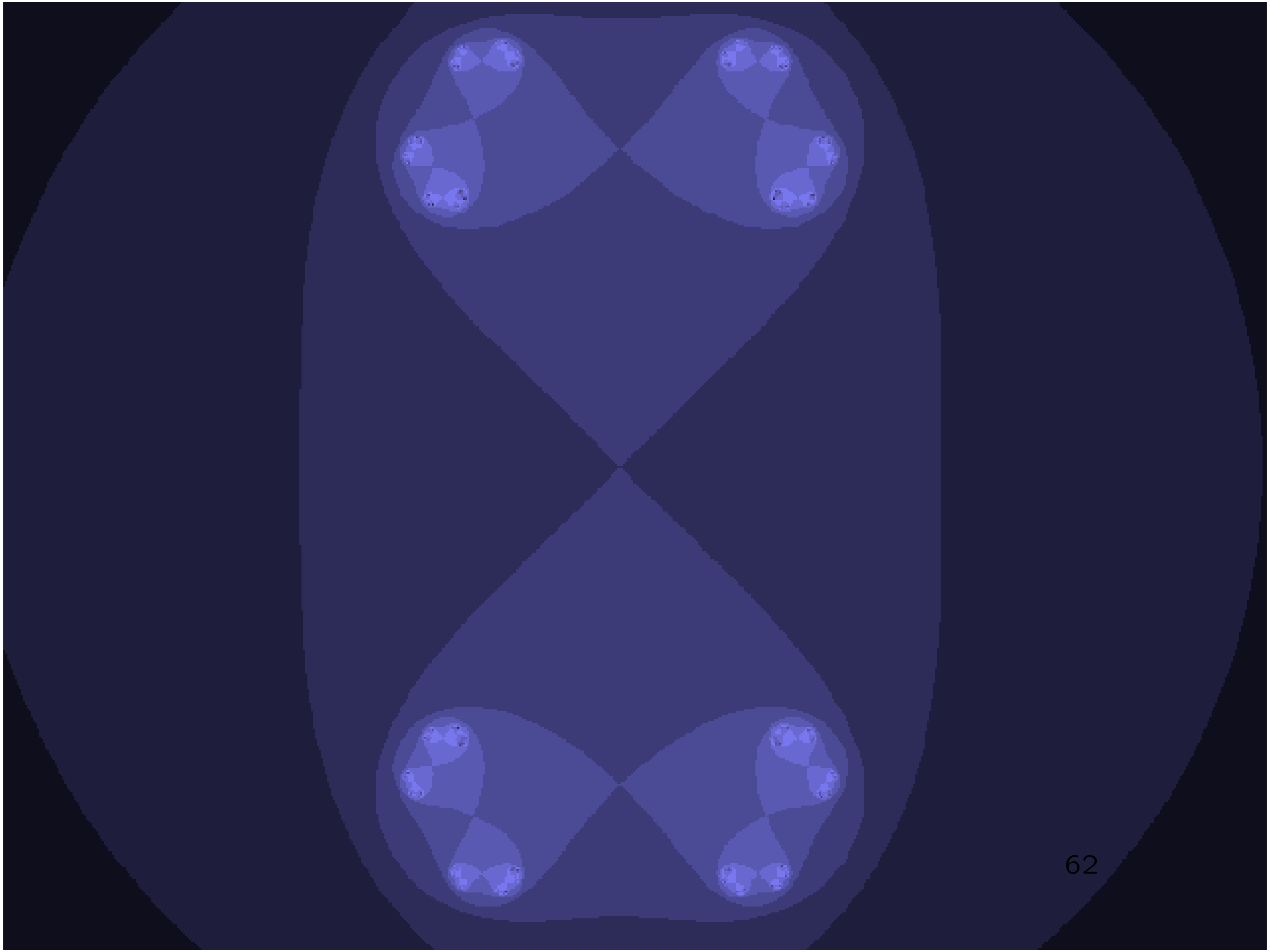
Dynamics decomposition

f induces a homeomorphism on the quotient space $M/\mathbf{Z}^{\perp}$.

Unusual view on holomorphic dynamics

Topologically invariant foliation in basins of attraction of superattracting points.

Example: $z^2 + 1$



Neutral entropy

A family of locally identity maps $f^{-n} \circ f^n$.

Their branches can be identified with the covering automorphisms group elements.

the sequence of nested groups of covering automorphisms

$$A_f \subset A_{f^2} \subset \cdots \subset A_{f^n} \subset \ldots,$$

Parially ordered abstract dynamical system with direction

A family of local identity maps $f^{-n} \circ f^n$.

Their branches can be identified with the covering automorphisms group elements.
The sequence of nested groups of covering automorphisms

$$A_f \subset A_{f^2} \subset \cdots \subset A_{f^n} \subset \dots,$$

Let A_{f^∞} be a direct limit group.

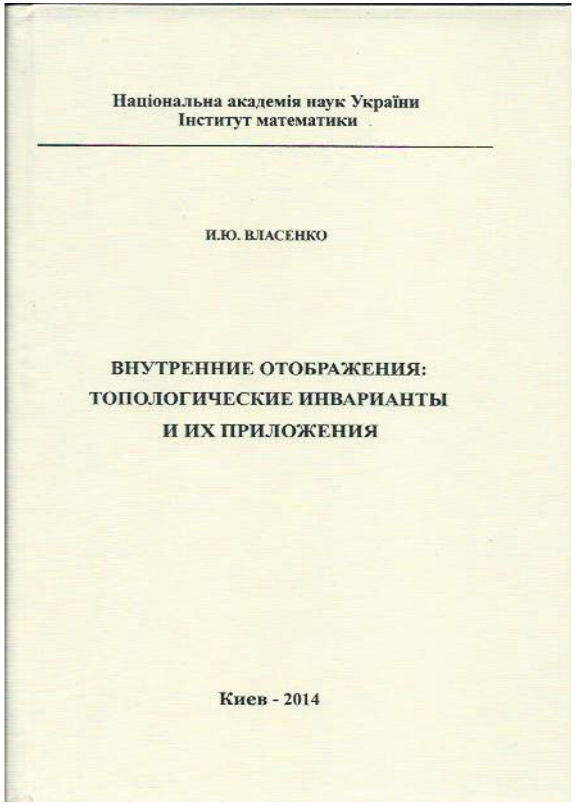
(M, A_{f^∞}) is a parially ordered abstract dynamical system with "ω" direction.
(We use $\perp$ here to distinguish from "ω" direction of f).

Евгений Алексеевич Барбашин (E. Barbashin)

Александр Максимович Стахи (A. Stakhi)

Neutral entropy

A_{f^∞} is an amenable group and it acts on the manifold with local identity maps $f^{-n} \circ f^n$.



Volume 1: general topological invariants and their properties, application to S^1 without using Milnor–Thurston kneading theory.

Volume 2: in process of writing.

Applying the theory to the inner mappings on surfaces.

Problems:

Topological classification of (some) wandering cylinders. wandering parts of basins of attraction, some polynomial maps.

A candidate class to be an analog of the Smale diffeomorphisms:

- the non-wandering set is hyperbolic.
- the wandering set is superwandering.
- branch points are regularly wandering superwandering.